MODULATIONAL INSTABILITY OF DOUBLE-PERIODIC WAVES
IN THE NONLINEAR SCHRÖDINGER EQUATION

DMITRY E. PELINOVSKY

Abstract. It is shown how to compute the modulational instability rates for the double-periodic solutions to the cubic NLS (nonlinear Schrödinger) equation by using the Lax linear equations. The wave function modulus of the double-periodic solutions is periodic both in space and time coordinates; such solutions generalize the standing waves which have the time-independent and space-periodic wave function modulus. Similar to other waves in the NLS equation, the double-periodic solutions are modulationally unstable and this instability is related to the bands of the Lax spectrum outside the imaginary axis. A simple numerical method is used to compute the modulational instability bands and to compare the instability rates of the double-periodic solutions with those of the standing periodic waves.

1. Introduction

Peregrine breather is a rogue wave arising on the background of the constant-amplitude wave due to its modulational instability [1, 2]. The focusing cubic NLS (nonlinear Schrödinger) equation is a canonical model which describes both the modulational instability and the formation of rogue waves. Formation of rogue waves on the constant-amplitude background have been modeled from different initial data such as local condensates [3], multi-soliton gases [4, 5, 6], and periodic perturbations [7, 8]. Rogue waves have been experimentally observed both in hydrodynamical and optical laboratories [9] (see recent reviews in [10, 11]).

Mathematical theory of rogue waves on the constant-amplitude background has seen many recent developments. Universal behavior of the modulationally unstable constant-amplitude background was studied asymptotically in [12, 13]. The finite-gap method was employed to relate the unstable modes on the constant-amplitude background with the occurrence of rogue waves [14, 15]. Rogue waves of infinite order were constructed in [16] based on recent developments in the inverse scattering method [17]. Rogue waves of the soliton superposition were studied asymptotically in the limit of many solitons [18, 19].

At the same time, rogue waves were also investigated on the background of standing periodic waves expressed by the Jacobian elliptic functions. Such exact solutions to the NLS equation were constructed first in [20] (see also early numerical work in [21] and the recent generalization in [22]). It is confirmed in [23] that these rogue waves arise due to the modulational instability of the standing periodic waves [24] (see also [25, 26]). Modulational instability of the periodic standing waves can be characterized by the separation of variables in the Lax pair of linear equations [27] (see also [28, 29]), compatibility of which gives the NLS equation. Modulational instability and rogue waves on the background of standing periodic waves have been experimentally observed in [30].

The main goal of this paper is to study the double-periodic solutions to the NLS equation, for which the wave function modulus is periodic with respect to both space and time coordinates. In
In particular, we consider two families of double-periodic solutions expressed as rational functions of the Jacobian elliptic functions which were constructed in the pioneering work [31]. These solutions represent perturbations of the Akhmediev breathers and describe generation of either phase-repeated or phase-alternating wave patterns [32, 33]. Rogue waves and modulational instability on the background of the double-periodic solutions were studied in [34] and [35] respectively (see also early numerical work in [36]). Experimental observation of the double-periodic solutions in optical fibers was reported in [37].

The double-periodic solutions constructed in [31] are particular cases of the quasi-periodic solutions of the NLS equation given by the Riemann Theta functions of genus two [38, 39, 40]. Rogue waves for general quasi-periodic solutions of any genus were considered in [41, 42, 43].

Here we show how to compute the modulational instability of the double-periodic solutions using the Floquet theory for the Lax linear equations both in space and time coordinates. The modulational instability rates of the double-periodic solutions are compared with those for the standing periodic waves. In order to provide a fair comparison, we normalize the amplitude of all solutions to unity. As a main outcome of this work, we show that the instability rates are larger for the constant-amplitude waves and smaller for the double-periodic waves.

The article is organized as follows. The explicit solutions to the NLS equation are reviewed in Section 2. Computations of the modulational instability rates for the standing periodic waves are reviewed in Section 3. New results on computations of the modulational instability rates for the double-periodic solutions are obtained in Section 4. Further directions are discussed in Section 5.

2. Explicit solutions to the NLS equation

We take the NLS equation in the standard form:

\[ i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0. \] (2.1)

This physical model has several symmetries:

- **translation**
  
  if \( \psi(x,t) \) is a solution, so is \( \psi(x + x_0, t + t_0) \), for every \( (x_0, t_0) \in \mathbb{R} \times \mathbb{R} \),

- **scaling**
  
  if \( \psi(x,t) \) is a solution, so is \( \alpha\psi(\alpha x, \alpha^2 t) \), for every \( \alpha \in \mathbb{R} \),

- **Lorentz transformation**
  
  if \( \psi(x,t) \) is a solution, so is \( \psi(x + \beta t, t)e^{-i\beta x - \frac{1}{2}\beta^2 t} \), for every \( \beta \in \mathbb{R} \).

In what follows, we use the scaling symmetry (2.3) to normalize the amplitude of periodic and double-periodic solutions to unity and the Lorentz symmetry (2.4) to set the wave speed to zero. We also neglect the translational parameters \( (x_0, t_0) \) due to the symmetry (2.2).

A solution \( \psi(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) to the NLS equation (2.1) is a compatibility condition of the following pair of linear Lax equations on \( \varphi(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2 \):

\[ \varphi_x = U(\lambda, \psi)\varphi, \quad U(\lambda, \psi) = \begin{pmatrix} \lambda & \psi \\ -\overline{\psi} & -\lambda \end{pmatrix} \] (2.5)
and
\[ \varphi_t = V(\lambda, \psi) \varphi, \quad V(\lambda, \psi) = i \left( \frac{\lambda^2 + \frac{1}{2} |\psi|^2}{\frac{1}{2} \bar{\psi}_x - \lambda \bar{\psi}} - \lambda^2 - \frac{1}{2} |\psi|^2 \right), \]  

(2.6)

where \( \bar{\psi} \) is the conjugate of \( \psi \) and \( \lambda \in \mathbb{C} \) is a spectral parameter.

The algebraic method developed in [34] allows us to construct the stationary (Lax–Novikov) equations which admit a large class of bounded periodic and quasi-periodic solutions to the NLS equation (2.1). The simplest first-order Lax–Novikov equation is given by

\[ \frac{du}{dx} + 2icu = 0, \]  

(2.7)

where \( c \) is arbitrary real parameter. A general solution \( u(x) = Ae^{-2icx} \) determines the constant-amplitude waves of the NLS equation (2.1) in the form:

\[ \psi(x, t) = Ae^{-2ic(x+ct)+iA^2t}, \]  

(2.8)

where \( A > 0 \) is the constant-amplitude parameter and translations in \( (x, t) \) are neglected due to the translational symmetry (2.2). Without loss of generality, \( c \) can be set to 0 due to the Lorentz transformation. Indeed, transformation (2.4) with \( \beta = -c \) transforms (2.8) to the equivalent form \( \psi(x, t) = Ae^{-icx-i\frac{1}{2}c^2t+iA^2t} \), which is obtained from \( \psi(x, t) = Ae^{-iA^2t} \) due to transformation (2.4) with \( \beta = c \). By the scaling transformation (2.3) with \( \alpha = A^{-1} \), the amplitude \( A \) can be set to unity.

The second-order Lax–Novikov equation is given by

\[ \frac{d^2u}{dx^2} + 2|u|^2u + 2ic\frac{du}{dx} - 4bu = 0, \]  

(2.9)

where \( (c, b) \) are arbitrary real parameters. Solutions \( u \) to the second-order equation (2.9) determines standing travelling waves of the NLS equation (2.1) in the form:

\[ \psi(x, t) = u(x + ct)e^{2ibt}. \]  

(2.10)

Without loss of generality, we set \( c = 0 \) due to the Lorentz transformation (2.4) with \( \beta = -c \). Waves with the trivial phase are of particular interest [20, 27]. There are two families of standing periodic waves with the trivial phase:

\[ \psi(x, t) = \text{dn}(x; k)e^{i(1-k^2)/2}t \]  

(2.11)

and

\[ \psi(x, t) = k\text{cn}(x; k)e^{i(k^2-1)/2}t, \]  

(2.12)

where the parameter \( k \in (0, 1) \) is the elliptic modulus. The solutions (2.11) and (2.12) are defined up to the scaling transformation (2.3) and translations in \( (x, t) \) (2.2). The amplitude (maximal value of \( |\psi| \)) is set to unity for (2.11) and to \( k \) for (2.12). In order to normalize the amplitude to unity for the cnoidal wave (2.12), we can use the scaling transformation (2.3) with \( \alpha = k^{-1} \).

Due to the well-known expansion formulas

\[ \text{dn}(x; k) = \text{sech}(x) + \frac{1}{4}(1-k^2)[\sinh(x)\cosh(x) + x]\tanh(x)\text{sech}(x) + \mathcal{O}((1-k^2)^2), \]  

\[ \text{cn}(x; k) = \text{sech}(x) - \frac{1}{4}(1-k^2)[\sinh(x)\cosh(x) - x]\tanh(x)\text{sech}(x) + \mathcal{O}((1-k^2)^2), \]
both the periodic waves (2.11) and (2.12) approaches the NLS soliton $\psi(x, t) = \text{sech}(x)e^{it/2}$ as $k \to 1$. In the other limit, the dnoidal periodic wave (2.11) approaches the constant-amplitude wave $\psi(x, t) = e^{it}$ as $k \to 0$, whereas the normalized cnoidal periodic wave (2.12) approaches the constant-amplitude wave $\psi(x, t) \sim \cos(x/k)e^{-it/(2k^2)}$ as $k \to 0$.

The third-order Lax–Novikov equation is given by

$$\frac{d^3u}{dx^3} + 6|u|^2\frac{du}{dx} + 2ic\left(\frac{d^2u}{dx^2} + 2|u|^2u\right) - 4b\frac{du}{dx} + 8iau = 0,$$

(2.13)

where $(a,b,c)$ are arbitrary real parameters. Waves with $a = c = 0$ are again of particular interest [31]. After a transformation of variables [34], such solutions can be written in the form:

$$\psi(x, t) = \left[Q(x, t) + i\delta(t)\right]e^{i\theta(t)},$$

(2.14)

where $Q(x, t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is periodic both in the space and time coordinates and $\delta(t) : \mathbb{R} \to \mathbb{R}$ has a double period in $t$ compared to $Q(x, t)$. There are two particular families of the double-periodic solutions (2.14), which can be written in the following forms (see Appendices A and B in [34]):

$$\psi(x, t) = k\frac{\text{cn}(t; k)\text{cn}(\sqrt{1+k}x; k) + i\sqrt{1+k}\text{sn}(t; k)\text{dn}(\sqrt{1+k}x; k)}{\sqrt{1+k}\text{dn}(\sqrt{1+k}x; k) - \text{dn}(t; k)\text{cn}(\sqrt{1+k}x; k)}e^{it}, \quad \kappa = \frac{1-k}{\sqrt{1+k}}$$

(2.15)

and

$$\psi(x, t) = \frac{\text{dn}(t; k)\text{cn}(\sqrt{2}x; k) + i\sqrt{k(1+k)}\text{sn}(t; k)}{\sqrt{1+k}\text{dn}(\sqrt{1+k}x; k)}e^{ikt}, \quad \kappa = \frac{1-k}{\sqrt{2}},$$

(2.16)

where $k \in (0, 1)$ is the elliptic modulus. The solutions (2.15) and (2.16) are defined up to the scaling transformation (2.3) and the translations in $(x, t)$ (2.2). The amplitude (maximal value of $|u|$) is $\sqrt{1+k} + 1$ for (2.15) and $\sqrt{1+k} + \sqrt{k}$ for (2.16). In order to normalize the amplitudes of the double-periodic waves to unity, we can use the scaling transformation (2.3) with $\alpha = (\sqrt{1+k} + 1)^{-1}$ and $\alpha = (\sqrt{1+k} + \sqrt{k})^{-1}$ respectively.

**Figure 1.** Amplitude-normalized double-periodic waves (2.15) (left) and (2.16) (right) with $k = 0.9$.

The double-periodic solutions (2.15) and (2.16) can be written in the form:

$$\psi(x, t) = \phi(x, t)e^{2ibt}, \quad \phi(x + L, t) = \phi(x, t + T) = \phi(x, t),$$

(2.17)
where $L > 0$ and $T > 0$ are fundamental periods in space and time coordinates, respectively, whereas $2b = 1$ for (2.15) and $2b = k$ for (2.16).

Figure 1 shows surface plots of $|\psi|$ on the $(x,t)$ plane within the fundamental periods. The amplitudes of the double-periodic waves on Fig. 1 have been normalized to unity by the scaling transformation (2.3). The solution (2.15) generates the phase-repeated wave patterns, whereas the solution (2.16) generates the phase-alternating patterns [32, 33, 34, 35].

As $k \to 1$, both the double-periodic solutions (2.15) and (2.16) approach to the same Akhmediev breather
\[
\psi(x,t) = \cos(\sqrt{2}x) + i\sqrt{2} \sinh(t) \sqrt{2} \cosh(t) - \cos(\sqrt{2}t)e^{it}.
\tag{2.18}
\]
As $k \to 0$, the solution (2.15) approaches the scaled NLS soliton
\[
\psi(x,t) = 2\text{sech}(2x)e^{2it},
\tag{2.19}
\]
whereas the solution (2.16) approaches the scaled cnoidal wave
\[
\psi(x,t) = \text{cn}(\sqrt{2}x; 1/\sqrt{2}).
\tag{2.20}
\]
These limits are useful to control accuracy of numerical computations of the modulational instability rate for the double-periodic solutions in comparison with the similar numerical computations for the standing waves.

3. Modulational instability of standing waves

Here we review how to use the linear equations (2.5)–(2.6) in order to study modulational instability of the standing periodic waves (2.10) (see [23, 27]). Separation of variables by
\[
\phi_1(x,t) = \chi_1(x+ct)e^{ibt+\Omega t}, \quad \phi_2(x,t) = \chi_2(x+ct)e^{-ibt+\Omega t},
\tag{3.1}
\]
where $\Omega \in \mathbb{C}$ is another spectral parameter, yields two spectral problems
\[
\chi_x = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix} \chi,
\tag{3.2}
\]
and
\[
\Omega \chi + c \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix} \chi = i \left( \lambda^2 + \frac{1}{2}|u|^2 - b + \frac{1}{2} du \frac{du}{dx} - \lambda \bar{u} - \lambda^2 - \frac{1}{2}|u|^2 + b \right) \chi.
\tag{3.3}
\]

We say that $\lambda$ belongs to the Lax spectrum of the spectral problem (3.2) if $\chi \in L^\infty(\mathbb{R})$. Since $u(x+L) = u(x)$ is periodic with the fundamental period $L > 0$, Floquet’s Theorem guarantees that bounded solutions of the linear equation (3.2) can be represented in the form:
\[
\chi(x) = \hat{\chi}(x)e^{i\theta x},
\tag{3.4}
\]
where $\hat{\chi}(x+L) = \hat{\chi}(x)$ and $\theta \in \left[-\frac{\pi}{L}, \frac{\pi}{L}\right]$. When $\theta = 0$ and $\theta = \pm \frac{\pi}{L}$, the bounded solutions (3.4) are periodic and anti-periodic, respectively.

Since the spectral problem (3.3) is a linear algebraic system, it admits a nonzero solution if and only if the determinant of the coefficient matrix is zero. The latter condition yields the $x$-independent relation between $\Omega$ and $\lambda$ in the form $\Omega^2 + P(\lambda) = 0$, where $P(\lambda)$ is given by
\[
P(\lambda) = \lambda^4 + 2ic\lambda^3 - (c^2 + 2b)\lambda^2 + 2i(a - bc)\lambda + b^2 - 2ac + 2d,
\tag{3.5}
\]
where parameters $a$ and $d$ appear in the two invariants of the second-order equation (2.9) given by
\[ \left| \frac{du}{dx} \right|^2 + |u|^4 - 4b|u|^2 = 8d \] (3.6)
and
\[ i \left( \frac{du}{dx} \bar{u} - \frac{d\bar{u}}{dx} \right) - 2c|u|^2 = 4a. \] (3.7)

Polynomial $P(\lambda)$ naturally occurs in the algebraic method [34]. For the standing waves of the trivial phase with $a = c = 0$, the polynomial $P(\lambda)$ can be written explicitly in the form:
\[ P(\lambda) = \lambda^4 - \frac{1}{2}(u_1^2 + u_2^2)\lambda^2 + \frac{1}{16}(u_1^2 - u_2^2)^2, \] (3.8)
where the turning points $u_1$ and $u_2$ parameterize $b$ and $d$ in the form:
\[ \begin{cases} 4b = u_1^2 + u_2^2, \\ 8d = -u_1^2u_2^2. \end{cases} \] (3.9)

Roots of $P(\lambda)$ are located at $(\lambda_1, -\lambda_1)$ and $(\lambda_2, -\lambda_2)$, where
\[ \lambda_1 = \frac{u_1 + u_2}{2}, \quad \lambda_2 = \frac{u_1 - u_2}{2}, \] (3.10)
so that the polynomial $P(\lambda)$ can be written in the factorized form:
\[ P(\lambda) = (\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2). \] (3.11)

By using perturbation to the standing waves (2.10) in the form
\[ \psi(x, t) = e^{2ibt} [u(x + ct) + v(x + ct, t)] \] (3.12)
and dropping the quadratic terms in $v$, we obtain the linearized system of equations which describe linear stability of the standing waves (2.10):
\[ \begin{cases} iv_t - 2bv + ivx + \frac{1}{2}v_{xx} + 2|u|^2v + u^2\bar{v} = 0, \\ -iv_t - 2b\bar{v} - iv\bar{x} + \frac{1}{2}\bar{v}_{xx} + 2|u|^2\bar{v} + \bar{u}^2v = 0. \end{cases} \] (3.13)

Separation of variables is given by
\[ v(x, t) = w_1(x)e^{i\Lambda}, \quad \bar{v}(x, t) = w_2(x)e^{i\Lambda}, \] (3.14)
where $w_1$ and $w_2$ are no longer complex conjugate to each other if $\Lambda \notin \mathbb{R}$. The spectral stability problem is defined by the set of admissible values of $\Lambda$ for which $(w_1, w_2) \in L^\infty(\mathbb{R})$. If $\lambda$ is in the Lax spectrum of the spectral problem (3.2), then the bounded squared eigenfunctions $\chi_1^2$ and $\chi_2^2$ determine the bounded eigenfunctions $w_1$ and $w_2$ of the linearized NLS equation (3.13) and $\Omega$ determines eigenvalues $\Lambda$, see (3.14), with the precise relation
\[ w_1 = \chi_1^2, \quad w_2 = -\chi_2^2, \quad \Lambda = 2\Omega. \] (3.15)

Validity of (3.15) can be checked directly from (3.2), (3.3), (3.13), and (3.14). If $\text{Re}(\Lambda) > 0$ for $\lambda$ in the Lax spectrum, the periodic standing wave (2.10) is called spectrally unstable. It is called modulationally unstable if the instability band intersects the origin in the $\Lambda$-plane.

It follows from (3.11) with either real $\lambda_1$, $\lambda_2$ or complex-conjugate $\lambda_1 = \bar{\lambda}_2$, if $\lambda \in i\mathbb{R}$ belongs to the Lax spectrum, then $\Lambda \in i\mathbb{R}$ in belongs to the stability spectrum. Thus, the spectral instability is only related to the Lax spectrum with $\lambda \notin i\mathbb{R}$.
For the dn-periodic wave (2.11) with $u_1 = 1$ and $u_2 = \sqrt{1-k^2}$, the amplitude is already normalized to unity and no scaling transformation is needed. Lax spectrum of the spectral problem (3.2) is shown on Fig. 2 (left) for $k = 0.9$. It follows from (3.11) that $P(\lambda) < 0$ for $\lambda \in (\lambda_2, \lambda_1)$ with $P(\lambda_{1,2}) = 0$, where $\lambda_{1,2}$ are given by (3.10). The unstable spectrum on the $\Lambda$-plane belongs to the finite segment on the real line which touches the origin as is shown on
Fig. 2 (right). It follows from (3.15) with \( \Omega = \pm i \sqrt{P(\lambda)} \) that

\[
\max_{\lambda \in [\lambda_2, \lambda_1]} \Lambda = \max_{\lambda \in [\lambda_2, \lambda_1]} 2 \sqrt{(\lambda_1^2 - \lambda^2)(\lambda^2 - \lambda_2^2)} = (\lambda_1^2 - \lambda_2^2) = \sqrt{1 - k^2}.
\]

Since the dn-periodic wave becomes the constant-amplitude wave of unit amplitude if \( k = 0 \), it is clear that the maximal instability rate is largest for the constant-amplitude wave with \( k = 0 \), monotonically decreasing in \( k \), and vanishes for the soliton limit \( k = 1 \). As \( k \to 1 \), the horizontal band on Fig. 2 (right) shrinks to an eigenvalue at the origin.

For the cn-periodic wave (2.12) with \( u_1 = k \) and \( u_2 = i \sqrt{1 - k^2} \), the amplitude is \( k \). Hence, we use the scaling transformation (2.3) with \( \alpha = k^{-1} \) in order to normalize the amplitude to unity. Lax spectrum of the spectral problem (3.2) for such an amplitude-normalized cn-periodic wave is shown on left panels of Fig. 3 for \( k = 0.85 \) (top) and \( k = 0.95 \) (bottom). The unstable spectrum on the \( \Lambda \)-plane is obtained from the same expressions \( \Lambda = \pm 2i \sqrt{P(\lambda)} \) when \( \lambda \) traverses along the bands of the Lax spectrum outside \( i \mathbb{R} \). The unstable spectrum is shown on the right panels of Fig. 3 for the same values of \( k \) and resembles the figure-eight band. The figure-eight band starts and ends at \( \Omega = 0 \) for \( \lambda = \lambda_1 \) and \( \lambda = \lambda_2 \). In both examples, the stability spectra are similar in spite of the differences in the Lax spectrum. The only difference is that the figure-eight band and the purely imaginary bands intersect for \( k = 0.85 \) (top) and do not intersect for \( k = 0.95 \) (bottom). As \( k \to 1 \), the figure-eight band shrinks to an eigenvalue at the origin.

![Figure 4](image.png)

**Figure 4.** Modulational instability rate \( \text{Re}(\Lambda) \) versus the Floquet parameter \( \theta \) for the amplitude-normalized dn-periodic (left) and cn-periodic (right) waves. The values of \( k \) in the elliptic functions are given in the plots.

Figure 4 compares the modulational instability growth rate for different standing waves of the same unit amplitude. \( \text{Re}(\Lambda) \) is plotted versus the Floquet parameter \( \theta \) in \([0, \frac{\pi}{L}]\) in (3.4). For the dn-periodic wave (left), we confirm that the growth rate is maximal for the constant-amplitude wave \( (k = 0) \) and is monotonically decreasing as \( k \) is increased in \((0, 1)\). For the cn-periodic wave (right), the growth rate is also maximal in the limit \( k \to 0 \), for which the amplitude-normalized cn-periodic wave is expanded as

\[
u(x) = \text{cn}(k^{-1}x; k) \sim 0.5e^{ikx} + 0.5e^{-ikx} \quad \text{as} \quad k \to 0.
\]

Due to the scaling transformation (2.3), the maximal growth rate in the limit \( k \to 0 \) is 0.25 instead of 1 and the Floquet parameter \( \theta \), for which it is attained, diverges to infinity as
$k \rightarrow 0$. As $k$ increases in $(0, 1)$, the growth rate becomes smaller and the Floquet parameter $\theta$ for which $\text{Re}(\Lambda) > 0$ moves towards the origin. The end point of the instability band is placed at the origin, when the bands of the Lax spectrum outside $i\mathbb{R}$ do not intersect $i\mathbb{R}$, see Fig. 3 (bottom).

4. MODULATIONAL INSTABILITY OF DOUBLE-PERIODIC WAVES

Here we describe the main result on how to study modulational instability of the double-periodic waves by using the linear equations (2.5)–(2.6). We write the solutions (2.15) and (2.16) in the form (2.17). Separation of variables by

$$\varphi_1(x, t) = \chi(x, t)e^{ibt+x\mu+it}, \quad \varphi_2(x, t) = \chi_2(x, t)e^{-ibt+x\mu+it},$$

(4.1)

where $\mu, \Omega \in \mathbb{C}$ are spectral parameters, yields two spectral problems

$$\chi_x + \mu \chi = \begin{pmatrix} \lambda & \phi \\ -\phi & -\lambda \end{pmatrix} \chi,$$

(4.2)

and

$$\chi_t + \Omega \chi = i \begin{pmatrix} \lambda^2 + \frac{1}{2} |\phi|^2 - b & \frac{1}{2} \frac{d\phi}{dx} + \lambda \phi \\ \frac{1}{2} \frac{d\phi}{dx} - \lambda \phi & -\lambda^2 - \frac{1}{2} |\phi|^2 + b \end{pmatrix} \chi.$$  

(4.3)

Parameters $\mu, \Omega \in \mathbb{C}$ are independent of $(x, t)$. This follows from the same compatibility of the linear Lax equations (2.5) and (2.6) if $\psi(x, t)$ in (2.17) satisfies the NLS equation (2.1).

By Floquet theorem, spectral parameters $\mu, \Omega \in \mathbb{C}$ are determined from the periodicity conditions $\chi(x + L, t) = \chi(x, t + T) = \chi(x, t)$ in terms of the spectral parameter $\lambda$. We distinguish between the space coordinate $x$ and the time coordinate $t$ in order to consider stability of the double-periodic waves (2.17) in the time evolution of the NLS equation (2.1).

The Lax spectrum is defined by the condition that $\lambda$ belongs to an admissible set for which the solution (4.1) is bounded in $x$. Hence $\mu = i\theta$ with real $\theta$ in $[-\frac{\pi}{T}, \frac{\pi}{T}]$ and $\lambda$ is computed from the spectral problem (4.2) with $\chi(x + L, t) = \chi(x, t)$ for every $t \in \mathbb{R}$.

With $\lambda$ defined in the Lax spectrum, the spectral problem (4.3) can be solved for the spectral parameter $\Omega$ under the condition that $\chi(x, t + T) = \chi(x, t)$ for every $x \in \mathbb{R}$. The corresponding solution to the linear system (4.2) and (4.3) generates the solution $v(x, t)$ of the linearized system (3.13) with $u \equiv \phi$ by means of the transformation formulas (3.14) and (3.15). Spectral parameter $\Omega$ is uniquely defined in the fundamental strip $\text{Im}(\Omega) \in [-\frac{\pi}{T}, \frac{\pi}{T}]$, while $\text{Re}(\Omega)$ determines the instability growth rate $\text{Re}(\Lambda)$ by $\Lambda = 2\Omega$.

If $\text{Re}(\Lambda) > 0$ for $\lambda$ in the Lax spectrum, the double-periodic wave (2.17) is called spectrally unstable. It is called modulationally unstable if the instability band intersects the origin in the $\Lambda$-plane. The amplitude-normalized double-periodic waves are taken by using the scaling transformation (2.3).

For the amplitude-normalized double-periodic wave (2.15) with $k = 0.85$ (top) and $k = 0.95$ (bottom), Fig. 5 shows the Lax spectrum of the spectral problem (4.2) with $\mu = i\theta$ and $\theta \in [-\frac{\pi}{T}, \frac{\pi}{T}]$ on the $\lambda$-plane (left) and the instability spectrum on the $\Lambda$-plane (right). The unstable spectrum is located at the boundary $\text{Im}(\Lambda) = \pm \frac{2\pi}{T}$ of the strip for every $k \in (0, 1)$. The double-periodic wave (2.15) is spectrally unstable but it is modulationally stable.

Fig. 6 shows the same as Fig. 5 but for the amplitude-normalized double-periodic wave (2.16) with $k = 0.3$ (top), $k = 0.6$ (middle), and $k = 0.9$ (bottom). The Lax spectrum on the $\lambda$-plane has three bands, two of which are connected either across the imaginary axis (top)
or across the real axis (middle and bottom), the third band is located on the real axis. The instability spectrum on the Λ-plane includes the figure-eight band and bands located near the boundary \( \text{Im}(Λ) = \pm \frac{2\pi}{T} \). As \( k \to 1 \), the figure-eight band becomes very thin and the stability spectra looks similar to the one on Fig. 5 because both the double-periodic solutions approach the Akhmediev breather (2.18).

Figure 5. Lax spectrum on the \( λ \)-plane (left) and stability spectrum on the \( Λ \)-plane (right) for the amplitude-normalized double-periodic wave (2.15) with \( k = 0.85 \) (top) and \( k = 0.95 \) (bottom).

Figure 7 compares the modulational instability growth rate for different double-periodic waves of the same unit amplitude. \( \text{Re}(Λ) \) is plotted versus the Floquet parameter \( θ \) in \( [0, \frac{π}{L}] \), where \( µ = iθ \) is defined in (4.2).

For the amplitude-normalized double-periodic wave (2.15) (left), the instability growth rate is maximal as \( k \to 1 \). The unstable band starts with the same cut-off value of \( θ \) and extends all the way to \( θ = \frac{π}{L} \). When \( k \to 0 \), the growth rates quickly decrease as the double-periodic wave approaches the NLS soliton (2.19).

For the amplitude-normalized double-periodic wave (2.16) (right), the maximal growth rate is large in the limit \( k \to 0 \), when the double-periodic wave is close to the particular cn-periodic wave (2.20). Then, the rates decrease when \( k \) is increased, however, the rates increase again and reach the maximal values as \( k \to 1 \) when the double-periodic wave approaches the Akhmediev breather (2.18).
Figure 6. The same as Figure 5 but for the amplitude-normalized double-periodic wave (2.16) with $k = 0.3$ (top), $k = 0.6$ (middle), and $k = 0.9$ (bottom).
Figure 7. Modulational instability rate $\text{Re}(\Lambda)$ versus the Floquet parameter $\theta$ for the amplitude-normalized double-periodic wave (2.15) (left) and the double-periodic wave (2.16) (right). The values of $k$ are given in the plots.

5. Conclusion

We have computed the modulational instability rates of the double-periodic waves of the NLS equation. By using the Lax pair of linear equations, we obtain the Lax spectrum with the Floquet theory in the spatial coordinate and the stability spectrum with the Floquet theory in the temporal coordinate. This separation of variables is computationally simpler than solving the full two-dimensional system of linearized NLS equations on the double-periodic solutions.

As the main outcome of the method, we have shown the modulational instability of the double-periodic solutions and have computed their instability rates, which are generally smaller compared to those for the standing periodic waves.

The concept can be extended to other double-periodic solutions of the NLS equation which satisfy the higher-order Lax–Novikov equations. Unfortunately, the other double-periodic solutions are only available in Riemann theta functions of genus $d \geq 2$, and for practical computations, one needs to first construct such double-periodic solutions numerically similarly to what was done in [41].

Acknowledgement: This work was supported in part by the National Natural Science Foundation of China (No. 11971103).

References

MODULATIONAL INSTABILITY OF DOUBLE-PERIODIC WAVES


Department of Mathematics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1