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Exponential and algebraic double-soliton solutions of the massive thirring model

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ABSTRACT

The newly discovered exponential and algebraic double-soliton solutions of the massive Thirring model in laboratory coordinates are placed in the context of the inverse scattering transform. We show that the exponential double-solitons correspond to double isolated eigenvalues in the Lax spectrum, whereas the algebraic double-solitons correspond to double embedded eigenvalues on the imaginary axis, where the continuous spectrum resides. This resolves the long-standing conjecture that multiple embedded eigenvalues may exist in the spectral problem associated with the massive Thirring model. To obtain the exponential double-solitons, we solve the Riemann–Hilbert problem with the reflectionless potential in the case of a quadruplet of double poles in each quadrant of the complex plane. To obtain the algebraic double-solitons, we consider the singular limit where the quadruplet of double poles degenerates into a symmetric pair of double embedded poles on the imaginary axis.

I. INTRODUCTION

We address the massive Thirring model (MTM) in laboratory coordinates, which can be written in the following normalized form

$$\begin{aligned} i(u_t+u_x)+v+|v|^2u=0,\\ i(v_t-v_x)+u+|u|^2v=0, \end{aligned} \tag{1.1}$$

where $u=u(x,t)$ and $v=v(x,t)$ are complex functions of real variables x and t. The MTM was introduced in Ref. 26 in the context of quantum field theory as a relativistically invariant nonlinear Dirac equation in one spatial dimension. It was found in Ref. 21 (see also Refs. 14, 16, and 22) that the MTM is a commutativity condition for a Lax pair of linear equations, hence it is completely integrable by the inverse scattering transform (IST) method. The Lax pair of linear equations for the MTM is given by

$$\partial_x\psi=L(u,v,\zeta)\psi,\quad \partial_t\psi=A(u,v,\zeta)\psi, \tag{1.2}$$

where $\zeta\in\mathbb{C}$ is the spectral parameter, $\psi=\psi(x,t)\in\mathbb{C}^2$ is the wave function, and the 2-by-2 matrices $L(u,v,\zeta)$ and $A(u,v,\zeta)$ are given by

$$L=\frac{i}{4}(|u|^2-|v|^2)\sigma_3-\frac{i}{2}\zeta\begin{pmatrix}0&\bar v\\ v&0\end{pmatrix}+\frac{i}{2\zeta}\begin{pmatrix}0&\bar u\\ u&0\end{pmatrix}+\frac{i}{4}(\zeta^2-\zeta^{-2})\sigma_3$$

and

$$A = -\frac{i}{4}\left(|u|^2 + |v|^2\right)\sigma_3 - \frac{i}{2}\zeta\begin{pmatrix}0 & \bar{v}\\ v & 0\end{pmatrix} - \frac{i}{2\zeta}\begin{pmatrix}0 & \bar{u}\\ u & 0\end{pmatrix} + \frac{i}{4}\left(\zeta^2 + \zeta^{-2}\right)\sigma_3.$$

Here the bar stands for the complex conjugation and $\sigma_3 = \mathrm{diag}(1,-1)$ is the third Pauli's matrix. The compatibility condition $\partial_t\partial_x\psi = \partial_x\partial_t\psi$ in the linear system (1.2) coincides with the MTM system (1.1).

The IST method based on the Riemann–Hilbert (RH) problem has been applied for the Lax pair (1.2) in the recent works Refs. 9 and 24 (see also earlier works Refs. 17 and 27). The IST method is used to obtain global solutions and to study the long-time dynamics of the MTM system (1.1) for the initial-value problem with the initial data $(u,v)|_{t=0} = (u_0, v_0)$ decaying to zero at infinity. The decay condition on (u_0, v_0) is required to be sufficiently fast so that the functions and their first and second derivatives are square integrable with the weight $\sqrt{1+x^2}$.[9] Exponential solitons satisfy this requirement and each soliton corresponds to a quadruplet of simple poles of the RH problem in each quadrant of the complex plane, or equivalently to simple isolated eigenvalues in the Lax spectrum of the linear system (1.2). However, algebraic solitons decay as $(u,v) = \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$ and hence they are not included in the IST method. Each algebraic soliton corresponds to a simple embedded eigenvalue in the Lax spectrum located on the imaginary axis (no embedded eigenvalues exist on the real axis).

The algebraic solitons in the MTM were studied in Ref. 15, where the perturbation theory for embedded eigenvalues in the Lax spectrum of the linear system (1.2) was developed. It was shown in (Ref. 15, Proposition 7.1) that a pair of simple embedded eigenvalues on the imaginary axis is structurally unstable and moves into a quadruplet of simple isolated eigenvalues in each quadrant of the complex plane under a generic perturbation of the initial data. A possibility of embedded eigenvalues of a higher algebraic multiplicity was also suggested in (Ref. 15, Lemma 6.4) with some precise conditions on the spatial decay of eigenvectors and generalized eigenvectors at infinity. Such embedded eigenvalues of higher algebraic multiplicity generally correspond to rational solutions of the MTM describing algebraic multi-solitons. However, the existence of such rational solutions has not been established in the literature up to very recently, despite many works on rational solutions in integrable systems (see, e.g., Refs. 6, 7, 10, 23, and 31–33).

Rational solutions of the MTM were constructed on the constant nonzero background in Refs. 5, 12, and 34. They are relevant to dynamics of rogue waves on the modulationally unstable background but do not describe the dynamics of algebraic solitons at the zero background. It was only recently shown in Ref. 8 (based on the Hirota's bilinear method developed in Ref. 4) that the algebraic double-solitons exist as the exact solutions of the MTM suggesting the existence of the higher-order algebraic solitons in a hierarchy of rational solutions to the MTM. Within the bilinear method, it was not shown in Ref. 8 that the algebraic double-solitons correspond to the double embedded eigenvalues in the Lax spectrum predicted in Ref. 15.

The main motivation for our work is to use the RH problem and to obtain the algebraic double-solitons of the MTM system (1.1) associated with the double embedded eigenvalues in the Lax spectrum of the linear system (1.2). To derive this result, we construct the exponential double-solitons associated with a quadruplet of double isolated eigenvalues in each quadrant of the complex plane and take the singular limit when the quadruplet of double isolated eigenvalues transforms into a symmetric pair of double embedded eigenvalues on the imaginary axis.

The study of double eigenvalues has started with the pioneering work,[36] where it was shown that the double eigenvalues of the associated spectral problem give the exponential double-solitons describing the slow (logarithmic in time) dynamics of two identical solitons of the focusing nonlinear Schrödinger (NLS) equation. Properties of such exponential double-solitons were recently studied in nonintegrable versions of the NLS equation in Ref. 20. The exponential double-solitons on the nonzero constant background were constructed in Ref. 25 after the development in the IST methods on the nonzero background in Ref. 3.

The double-soliton solutions in the closely related derivative NLS equation were constructed by using the Darboux transformations in Refs. 11, 30, and 35. It was understood in Ref. 35 that the algebraic double-solitons arise from the exponential double-solitons in the singular limit, for which the modified Darboux transformations have been developed in Ref. 11. The IST method was also employed in the context of the derivative NLS equation to construct the exponential double-solitons from the double poles of the RH problem in Refs. 28 and 37–39. Algebraic solitons of the derivative NLS equation and closely related equations were further considered in Ref. 18 and 29.

Although both the derivative NLS equation and the MTM system in characteristic coordinates are related to the same spectral problem,[13,14] the computational details for the MTM system in laboratory coordinates are different and technically more complicated. We close this gap in the literature by presenting the exponential double-solitons of the MTM system (1.1) for the double isolated eigenvalues of the linear system (1.2), see also Ref. 19 for the very recent work on exponential multi-solitons. *The main application of this result is to obtain the algebraic double-solitons and the double embedded eigenvalues in the singular limit, where the RH problem cannot be used.*

For the spectral problem associated with the focusing NLS equation on a nonzero background, it was understood in Ref. 1 how to modify the RH problem for the simple and multiple embedded eigenvalues at the end points of the continuous spectrum in order to construct the rogue waves of high multiplicity.[2] This modification of the RH problem has not been developed so far for the spectral problem associated with the derivative NLS equation and the MTM system on the zero background. It is still unclear how the simple or multiple embedded eigenvalues can be constructed in the RH problem directly. We hope that our work will motivate further study of the associated spectral problems with embedded eigenvalues.

This paper is organized as follows. Section II introduces the RH problem for the MTM and formulates the main results. The exponential double-solitons are constructed in Sec. III from the isolated double-pole solutions of the RH problem. The algebraic double-solitons are obtained in Sec. IV by taking the singular limit to the embedded double-pole solutions of the RH problem. Appendix A reports similar computations for the exponential and algebraic single-solitons for convenience of readers. Appendix B reviews the construction of the exponential double-solitons in the MTM system by using the bilinear Hirota method.

II. RH PROBLEM FOR MTM AND MAIN RESULTS

Assume that $(u,v) \to (0,0)$ as $|x| \to \infty$ fast enough, see Lemmas 2.1 and 2.2 below for precise requirements on (u,v). We define the matrix Jost functions for the linear system (1.2) from the boundary conditions:

$$\psi^{(\pm)}(\zeta,x,t) \to \begin{pmatrix} e^{\frac{i}{4}(\zeta^2-\zeta^{-2})x+\frac{i}{4}(\zeta^2+\zeta^{-2})t} & 0 \\ 0 & e^{-\frac{i}{4}(\zeta^2-\zeta^{-2})x-\frac{i}{4}(\zeta^2+\zeta^{-2})t} \end{pmatrix} \quad \text{as} \quad x \to \pm\infty. \tag{2.1}$$

For simplicity of notations, we will drop the dependence of $\psi^{(\pm)}$ on (x,t). Since the Jost functions $\psi^{(\pm)}(\zeta)$ represent the fundamental matrix solutions of the linear system (1.2), they are related to each other by the scattering relations introduced for $\zeta \in (\mathbb{R}\cup i\mathbb{R})\backslash\{0\}$ as

$$\psi^{(-)}(\zeta) = \psi^{(+)}(\zeta)\begin{pmatrix} \overline{a}(\zeta) & b(\zeta) \\ -\overline{b}(\zeta) & a(\zeta) \end{pmatrix}, \tag{2.2}$$

where the symmetry of scattering coefficients $a(\zeta)$ and $b(\zeta)$ follows from the symmetry of matrix Jost functions:

$$\psi^{(\pm)}(\zeta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\overline{\psi}^{(\pm)}(\zeta)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.3}$$

As is explained in Ref. 24, the linear system (1.2) can be folded to the squared spectral parameter $\lambda := \zeta^2$ in two different ways, one is suitable near $\zeta = 0$ and the other one is suitable near $\zeta = \infty$. Following,[9] we will only consider the second transformation, from which we will define the Riemann–Hilbert (RH) problem and solve it for the exponential double-solitons, see Theorem 2.1 below.

Hence we introduce the modified Jost functions as

$$\begin{cases} n_1^{(\pm)}(\lambda) := T(v,\zeta)\psi_1^{(\pm)}(\zeta)e^{-\frac{i}{4}(\zeta^2-\zeta^{-2})x-\frac{i}{4}(\zeta^2+\zeta^{-2})t}, \\ n_2^{(\pm)}(\lambda) := \zeta^{-1}T(v,\zeta)\psi_2^{(\pm)}(\zeta)e^{\frac{i}{4}(\zeta^2-\zeta^{-2})x+\frac{i}{4}(\zeta^2+\zeta^{-2})t}, \end{cases} \tag{2.4}$$

where the subscripts indicate the columns of the 2-by-2 matrices and the transformation matrix is given by

$$T(v,\zeta) := \begin{pmatrix} 1 & 0 \\ v & \zeta \end{pmatrix}.$$

It follows from (2.1) that the modified Jost functions satisfy

$$n_1^{(\pm)}(\lambda) \to e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n_2^{(\pm)}(\lambda) \to e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{as} \quad x \to \pm\infty.$$

Moreover, $n_{1,2}^{(\pm)}(\lambda)$ satisfy the integral equations, from which the following properties were proven in (Ref. 24, Lemmas 3–5).

Lemma 2.1. Let $(u,v) \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ and $(u_x,v_x) \in L^1(\mathbb{R})$. For every $\lambda \in \mathbb{R}\backslash\{0\}$, there exist unique bounded Jost functions $n_1^{(\pm)}(\lambda)$ and $n_2^{(\pm)}(\lambda)$. For every $x \in \mathbb{R}$, $n_1^{(\pm)}$ and $n_2^{(\pm)}$ are continued analytically in $\mathbb{C}^\pm$ and satisfy the following limits as $|\lambda| \to \infty$ and $\lambda \to 0$ along a contour in the domains of their analyticity:

$$\lim_{|\lambda|\to\infty} \frac{n_1^{(\pm)}(\lambda)}{n_1^{\pm\infty}} = e_1, \quad \lim_{|\lambda|\to\infty} \frac{n_2^{(\pm)}(\lambda)}{n_2^{\pm\infty}} = e_2, \tag{2.5}$$

and

$$\lim_{\lambda\to 0}\left[n_1^{\pm\infty} n_1^{(\pm)}(\lambda)\right] = e_1 + ve_2, \quad \lim_{\lambda\to 0}\left[n_2^{\pm\infty} n_2^{(\pm)}(\lambda)\right] = \bar{u}e_1 + (1+\bar{u}v)e_2, \tag{2.6}$$

where

$$n_1^{\pm\infty} := e^{\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}, \quad n_2^{\pm\infty} := e^{-\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}.$$

Recall that $\lambda := \zeta^2$ and that $\lambda \in \mathbb{R}$ for $\zeta \in (\mathbb{R}\cup i\mathbb{R})$. Hence we define new scattering coefficients for $\lambda \in \mathbb{R}\backslash\{0\}$ as

$$\alpha(\lambda) := a(\zeta), \quad \beta_+(\lambda) := \zeta b(\zeta), \quad \beta_-(\lambda) := \zeta^{-1}b(\zeta).$$

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After the folding transformation (2.4), the scattering relations (2.2) are modified as follows

$$n^{(-)}(\lambda) = n^{(+)}(\lambda)\begin{pmatrix} \overline{\alpha}(\lambda) & \beta_-(\lambda)e^{2i\theta(\lambda)} \\ -\overline{\beta}_+(\lambda)e^{-2i\theta(\lambda)} & \alpha(\lambda) \end{pmatrix}, \tag{2.7}$$

where

$$\theta(\lambda) := \frac{1}{4}(\lambda - \lambda^{-1})x + \frac{1}{4}(\lambda + \lambda^{-1})t. \tag{2.8}$$

The following lemma was proven in (Ref. 24, Lemma 6).

Lemma 2.2. Let $(u, v) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $(u_x, v_x) \in L^1(\mathbb{R})$. Then, α is continued analytically into $\mathbb{C}^+$ with the following limits in $\mathbb{C}^+$:

$$\lim_{|\lambda|\to\infty} \alpha(\lambda) = e^{-\frac{i}{4}\int_{\mathbb{R}}(|u|^2+|v|^2)dy} \tag{2.9}$$

and

$$\lim_{\lambda\to 0} \alpha(\lambda) = e^{\frac{i}{4}\int_{\mathbb{R}}(|u|^2+|v|^2)dy}, \tag{2.10}$$

whereas $\beta_\pm$ are not continued analytically outside $\mathbb{R}$ and satisfy the limits

$$\lim_{|\lambda|\to\infty} \beta_\pm(\lambda) = \lim_{\lambda\to 0} \beta_\pm(\lambda) = 0.$$

The RH problem for the modified Jost functions $n^{(\pm)}(\lambda)$ is constructed as follows. We first define the sectionally meromorphic matrix $P(\lambda) \in \mathbb{C}^{2\times 2}$ by

$$P(\lambda) := \begin{cases} \left(n_1^{(+)}(\lambda), \dfrac{n_2^{(-)}(\lambda)}{\alpha(\lambda)}\right), & \lambda \in \mathbb{C}^+, \\ \left(\dfrac{n_1^{(-)}(\lambda)}{\bar{\alpha}(\lambda)}, n_2^{(+)}(\lambda)\right), & \lambda \in \mathbb{C}^-. \end{cases} \tag{2.11}$$

By using (2.5) and (2.9), we obtain the following limits as $|\lambda| \to \infty$ in the domain of meromorphicity of $P(\lambda)$:

$$\lim_{|\lambda|\to\infty} P(\lambda) = \begin{pmatrix} n_1^{+\infty} & 0 \\ 0 & n_2^{+\infty} \end{pmatrix} =: P^\infty, \tag{2.12}$$

where $\overline{P^\infty} = (P^\infty)^{-1}$. We finally define a complex-valued function $M(\lambda) := (P^\infty)^{-1}P(\lambda)$ which satisfies the normalized RH problem with the following three properties:

- $M(\lambda)$ is meromorphic in $\mathbb{C}\backslash\mathbb{R}$, with finitely many poles at $\{\lambda_1, \ldots, \lambda_n\} \in \mathbb{C}^+$ and $\{\bar{\lambda}_1, \ldots, \bar{\lambda}_n\} \in \mathbb{C}^-$ and specific normalization of the principal part of Laurent expansions.
- $M(\lambda) \to \mathbb{I}$ as $|\lambda| \to \infty$, where $\mathbb{I}$ is the 2-by-2 identity matrix.
- $M(\lambda)$ is continuous on both sides of $\mathbb{R}$ with $M_\pm(\lambda) := \lim_{\mathrm{Im}(\lambda)\to\pm 0} M(\lambda)$ satisfying

  $$M_+(\lambda) = M_-(\lambda)V(\lambda), \quad \lambda \in \mathbb{R},$$

  where

  $$V(\lambda) := \begin{pmatrix} 1 & -r_-(\lambda)e^{2i\theta(\lambda)} \\ -\bar{r}_+(\lambda)e^{-2i\theta(\lambda)} & 1 + \bar{r}_+(\lambda)r_-(\lambda) \end{pmatrix}, \quad r_\pm(\lambda) := \frac{\beta_\pm(\lambda)}{\alpha(\lambda)}.$$

It follows from (2.6) and (2.10) [see also (Ref. 9, Proposition 2.24)] that the potentials (u, v) for solutions of the MTM system (1.1) can be recovered from solutions of the RH problem by using the following asymptotic limits taken in the domains of meromorphicity of $M(\lambda)$:

$$u = \lim_{\lambda\to 0} \overline{M}_{12}(\lambda), \quad v = \lim_{\lambda\to 0} M_{21}(\lambda). \tag{2.13}$$

Solvability of the RH problem under some conditions of the reflection coefficients $r_\pm(\lambda)$ was studied in Refs. 9 and 24. In this work, we consider the reflectionless case $r_\pm(\lambda) \equiv 0$ for $\lambda \in \mathbb{R}$ in the particular case when $M(\lambda)$ admits a pair of double poles at $\lambda_0 \in \mathbb{C}^+$ and $\bar{\lambda}_0 \in \mathbb{C}^-$ due to symmetry (2.3).

It is well-known (see, e.g., Ref. 9) that a pair of simple poles of $M(\lambda)$ leads to a single-soliton solution. For completeness, we give details of the RH problem with a pair of simple poles of $M(\lambda)$ in Appendix A. To simplify the presentation of soliton solutions, we should use the basic symmetries of the MTM system. In particular, the relativistically invariant MTM system (1.1) admits the Lorentz symmetry

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} \mapsto \begin{bmatrix} \left(\frac{1-c}{1+c}\right)^{1/4} u\left(\frac{x+ct}{\sqrt{1-c^2}}, \frac{t+cx}{\sqrt{1-c^2}}\right) \\ \left(\frac{1+c}{1-c}\right)^{1/4} v\left(\frac{x+ct}{\sqrt{1-c^2}}, \frac{t+cx}{\sqrt{1-c^2}}\right) \end{bmatrix}, \qquad c \in (-1,1). \tag{2.14}$$

In addition, it admits the translational and rotational symmetries

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} \mapsto \begin{bmatrix} u(x+x_0, t+t_0) e^{i\theta_0} \\ v(x+x_0, t+t_0) e^{i\theta_0} \end{bmatrix}, \qquad x_0, t_0, \theta_0 \in \mathbb{R}. \tag{2.15}$$

By using (2.14) and (2.15), the single-soliton solutions can be expressed in a short form:

$$\begin{cases} u(x,t) = i(\sin\gamma)\mathrm{sech}\left(x\sin\gamma - i\frac{\gamma}{2}\right)e^{-it\cos\gamma}, \\ v(x,t) = -i(\sin\gamma)\mathrm{sech}\left(x\sin\gamma + i\frac{\gamma}{2}\right)e^{-it\cos\gamma}, \end{cases} \tag{2.16}$$

where $\gamma \in (0,\pi)$ is a free parameter. More general single-soliton solutions can be extended with speed parameter $c \in (-1,1)$ by using (2.14) and with two translational parameters $x_0, t_0 \in \mathbb{R}$ by using (2.15), where translation in θ_0 is linearly dependent from translation in t_0.

The normalized single-soliton solution (2.16) corresponds to a pair of simple poles of $M(\lambda)$ at $\lambda_0 = e^{i\gamma} \in \mathbb{C}^+$ and $\bar{\lambda}_0 = e^{-i\gamma} \in \mathbb{C}^-$ with $\gamma \in (0,\pi)$, see Appendix A. The double-soliton solutions will also be constructed for a pair of double poles of $M(\lambda)$ at $\lambda_0 = e^{i\gamma} \in \mathbb{C}^+$ and $\bar{\lambda}_0 = e^{-i\gamma} \in \mathbb{C}^-$.

The following theorem gives the explicit representation of the double-soliton solutions. As we show in Appendix B, this representation coincides with the explicit formula obtained by the bilinear Hirota method developed in Ref. 4.

Theorem 2.1. *Let $\lambda_0 = e^{i\gamma}$ and $\bar{\lambda}_0 = e^{-i\gamma}$ with $\gamma \in (0,\pi)$ be a pair of double poles of $M(\lambda)$ in the RH problem. Then, the solution (u,v) of the MTM system (1.1) obtained from (2.13) is given by*

$$u = \frac{\bar{N}_u}{\bar{D}}, \quad v = \frac{N_v}{D}, \tag{2.17}$$

where

$$N_u = 4i(\sin\gamma)^2 e^{-x\sin\gamma + it\cos\gamma - \frac{i\gamma}{2}}\left((x-\tilde{x}_0)\cos\gamma + i(t-\tilde{t}_0)\sin\gamma + i - e^{-2x\sin\gamma - i\gamma}[2\cot\gamma + (x-\tilde{x}_0)\cos\gamma - i(t-\tilde{t}_0)\sin\gamma]\right),$$

$$N_v = 4i(\sin\gamma)^2 e^{-x\sin\gamma - it\cos\gamma - \frac{i\gamma}{2}}\left((x-\tilde{x}_0)\cos\gamma - i(t-\tilde{t}_0)\sin\gamma - e^{-2x\sin\gamma - i\gamma}[2\cot\gamma + (x-\tilde{x}_0)\cos\gamma + i(t-\tilde{t}_0)\sin\gamma + i]\right),$$

and

$$D = 1 + e^{-4x\sin\gamma - 2i\gamma} + 2e^{-2x\sin\gamma - i\gamma}\left(1 + 2(\sin\gamma)^2\left[\cot\gamma + (x-\tilde{x}_0)\cos\gamma + \frac{i}{2}\right]^2 + 2(\sin\gamma)^4\left[t - \tilde{t}_0 + \frac{1}{2\sin\gamma}\right]^2\right),$$

where $\tilde{x}_0, \tilde{t}_0 \in \mathbb{R}$ are arbitrary parameters in addition to arbitrary parameters $c \in (-1,1)$ and $x_0, t_0 \in \mathbb{R}$, which are obtained from the transformations (2.14) and (2.15).

Remark 2.1. Parameter $\tilde{t}_0$ is trivially removed by using translational symmetries (2.15) with translations in θ_0 and t_0. Hence, the double-soliton solutions of Theorem 2.1 only have two non-trivial parameters: $\gamma \in (0,\pi)$ and $\tilde{x}_0 \in \mathbb{R}$.

Although the explicit form of double-soliton solutions in Theorem 2.1 can be obtained by algebraic methods such as Darboux transformations or the bilinear Hirota method, see Appendix B, the RH problem enables us to clarify the Lax spectrum of the double-soliton solutions, in particular, the existence of a quadruplet $\{\zeta_0, \bar{\zeta}_0, -\zeta_0, -\bar{\zeta}_0\}$ of double eigenvalues, where each double eigenvalue is identified according to the following definition.

Definition 2.1. We say that $\zeta_0 \in \mathbb{C}$ is a double eigenvalue of the linear system (1.2) if there exists an eigenvector $\psi_0 \in H^1(\mathbb{R},\mathbb{C}^2)$ and a generalized eigenvector $\psi_1 \in H^1(\mathbb{R},\mathbb{C}^2)$ satisfying the following linear equations:

$$\partial_x\psi_0 = L(u,v,\zeta_0)\psi_0, \quad \partial_t\psi_0 = A(u,v,\zeta_0)\psi_0 \tag{2.18}$$

and

$$\partial_x \psi_1 = L(u,v,\zeta_0)\psi_1 + \partial_\zeta L(u,v,\zeta_0)\psi_0, \quad \partial_t \psi_1 = A(u,v,\zeta_0)\psi_1 + \partial_\zeta A(u,v,\zeta_0)\psi_0, \tag{2.19}$$

and there exists no second generalized eigenvector $\psi_2 \in H^1(\mathbb{R},\mathbb{C}^2)$ *satisfying the linear equation*

$$\partial_x \psi_2 = L(u,v,\zeta_0)\psi_2 + \partial_\zeta L(u,v,\zeta_0)\psi_1 + \frac{1}{2}\partial_\zeta^2 L(u,v,\zeta_0)\psi_0,$$
$$\partial_t \psi_2 = A(u,v,\zeta_0)\psi_2 + \partial_\zeta A(u,v,\zeta_0)\psi_1 + \frac{1}{2}\partial_\zeta^2 A(u,v,\zeta_0)\psi_0.$$

Furthermore, we say that $\zeta_0 \in \mathbb{C}$ *is a simple eigenvalue if there exists an eigenvector* $\psi_0 \in H^1(\mathbb{R},\mathbb{C}^2)$ *satisfying (2.18) but there exists no generalized eigenvector* $\psi_1 \in H^1(\mathbb{R},\mathbb{C}^2)$ *satisfying (2.19).*

Remark 2.2. Since the spectral problem $\partial_x \psi_0 = L(u,v,\zeta_0)\psi_0$ *is a second-order differential equation with the zero-trace matrix, it is clear that there exists at most one linearly independent eigenvector* $\psi_0 \in H^1(\mathbb{R},\mathbb{C}^2)$ *decaying to zero as* $|x| \to \infty$*. However, no generalized eigenvector* $\psi_1 \in H^1(\mathbb{R},\mathbb{C}^2)$ *exists for a simple eigenvalue* $\zeta_0 \in \mathbb{C}$ *because a Fredholm solvability condition is not satisfied for the linear inhomogeneous equation* $\partial_x \psi_1 = L(u,v,\zeta_0)\psi_1 + \partial_\zeta L(u,v,\zeta_0)\psi_0$*. If* $\zeta_0 \in \mathbb{C}$ *is a double eigenvalue, there exists the generalized eigenvector* $\psi_1 \in H^1(\mathbb{R},\mathbb{C}^2)$ *of the linear equation* $\partial_x \psi_1 = L(u,v,\zeta_0)\psi_1 + \partial_\zeta L(u,v,\zeta_0)\psi_0$ *but the Fredholm solvability condition is not satisfied for*

$$\partial_x \psi_2 = L(u,v,\zeta_0)\psi_2 + \partial_\zeta L(u,v,\zeta_0)\psi_1 + \frac{1}{2}\partial_\zeta^2 L(u,v,\zeta_0)\psi_0$$

so that no second generalized eigenvector $\psi_2 \in H^1(\mathbb{R},\mathbb{C}^2)$ *exists.*

Based on the solution in Section III, we prove that if (u,v) is given by (2.17), then $\zeta_0 := \sqrt{\lambda_0} = e^{\frac{i}{2}\gamma}$ is a double eigenvalue of the linear system (1.2) in the sense of Definition 2.1. The knowledge of eigenvectors and generalized eigenvectors satisfying (2.18) and (2.19) is particularly important when the exponential double-soliton solution of Theorem 2.1 converges as $\gamma \to \pi$ to the algebraic double-soliton solution obtained in Ref. 8. The following theorem states that the corresponding Lax spectrum includes the double embedded eigenvalue $\zeta_0 = i$ of the linear system (1.2) with only one eigenvector $\psi_0 \in H^1(\mathbb{R},\mathbb{C}^2)$ and one generalized eigenvector $\psi_1 \in H^1(\mathbb{R},\mathbb{C}^2)$ satisfying (2.18) and (2.19) with $\zeta_0 = i$.

Theorem 2.2. *Let* $\lambda_0 = e^{i\gamma}$ *and* $\bar{\lambda}_0 = e^{-i\gamma}$ *with* $\gamma \in (0,\pi)$ *be a pair of double poles of* $M(\lambda)$ *with the solution* (u,v) *of the MTM system (1.1) obtained in Theorem 2.1. With a proper choice of* $\tilde{x}_0$ *and* $\tilde{t}_0$*, this solution transforms in the limit* $\gamma \to \pi$ *to the form:*

$$u_{\text{alg}}(x,t) = \frac{-\frac{8}{3}x^3 - 4ix^2 + 2x - i - 4i(t-\tilde{t}_0)(i+2x) + 8\tilde{x}_0}{\frac{4}{3}x^4 + \frac{8}{3}ix^3 + 2x^2 + 2ix - \frac{1}{4} - 4(t-\tilde{t}_0)^2 + 4\tilde{x}_0(i+2x)} e^{it} \tag{2.20}$$

and

$$v_{\text{alg}}(x,t) = \frac{-\frac{8}{3}x^3 + 4ix^2 + 2x + i + 4i(t-\tilde{t}_0)(i-2x) + 8\tilde{x}_0}{\frac{4}{3}x^4 - \frac{8}{3}ix^3 + 2x^2 - 2ix - \frac{1}{4} - 4(t-\tilde{t}_0)^2 - 4\tilde{x}_0(i-2x)} e^{it}, \tag{2.21}$$

where $\tilde{x}_0, \tilde{t}_0 \in \mathbb{R}$ *are (new) arbitrary parameters in addition to arbitrary parameters* $c \in (-1,1)$ *and* $x_0, t_0 \in \mathbb{R}$*, which are obtained from the transformations (2.14) and (2.15). The linear equations (2.18) and (2.19) with* $(u,v) = (u_{\text{alg}}, v_{\text{alg}})$ *and* $\zeta_0 = i$ *admit the eigenvector*

$$\psi_0 = e^{-\frac{i}{2}t} T^{-1} \mathfrak{n}_0, \tag{2.22}$$

where $T^{-1} := \left[T(v_{alg}, i)\right]^{-1} = \begin{pmatrix} 1 & 0 \\ iv_{alg} & -i \end{pmatrix}$ *and* $\mathfrak{n}_0 = (\mathfrak{n}_{01}, \mathfrak{n}_{02})^T$ *is given by*

$$\mathfrak{n}_{01} = e^{\frac{i}{4}\int_{+\infty}^x (|u|^2+|v|^2)dy} \frac{2x^2 - 2i(t-\tilde{t}_0) + \frac{1}{2}}{\frac{4}{3}x^4 - \frac{8}{3}ix^3 + 2x^2 - 2ix - \frac{1}{4} - 4(t-\tilde{t}_0)^2 - 4\tilde{x}_0(i-2x)},$$
$$\mathfrak{n}_{02} = e^{-\frac{i}{4}\int_{+\infty}^x (|u|^2+|v|^2)dy + it} \frac{-2ix^2 - 4x + 2(t-\tilde{t}_0) + \frac{3}{2}i}{\frac{4}{3}x^4 - \frac{8}{3}ix^3 + 2x^2 - 2ix - \frac{1}{4} - 4(t-\tilde{t}_0)^2 - 4\tilde{x}_0(i-2x)} e^{it}.$$

and the generalized eigenvector

$$\psi_1 = 2ie^{-\frac{i}{2}t} T^{-1} \mathfrak{n}_1 + i\partial_\zeta T(v,\zeta)\psi_0, \tag{2.23}$$

where $\partial_\zeta T(v,\zeta) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\mathfrak{n}_1 = (\mathfrak{n}_{11}, \mathfrak{n}_{12})^T$ *is given by*

$$\mathfrak{n}_{11} = e^{\frac{i}{4}\int_{+\infty}^x (|u|^2+|v|^2)dy} \frac{-\frac{1}{3}ix^3 - 2x^2 - \frac{4}{3}ix + x(t-\tilde{t}_0) + 2i(t-\tilde{t}_0) - \frac{1}{2} - 2i\tilde{x}_0}{\frac{4}{3}x^4 - \frac{8}{3}ix^3 + 2x^2 - 2ix - \frac{1}{4} - 4(t-\tilde{t}_0)^2 - 4\tilde{x}_0(i-2x)},$$

$$\mathfrak{n}_{12} = e^{-\frac{i}{4}\int_{+\infty}^x (|u|^2+|v|^2)dy+it} \frac{-\frac{1}{3}x^3 - ix^2 - \frac{15}{4}x + ix(t-\tilde{t}_0) + \frac{5}{4}i + 3(t-\tilde{t}_0) - 2\tilde{x}_0}{\frac{4}{3}x^4 - \frac{8}{3}ix^3 + 2x^2 - 2ix - \frac{1}{4} - 4(t-\tilde{t}_0)^2 - 4\tilde{x}_0(i-2x)} e^{it}.$$

Remark 2.3. The eigenvector ψ_0 *and generalized eigenvector* ψ_1 *in (2.22) and (2.23) for the double embedded eigenvalue* $\zeta = i$ *satisfy the criterion for the spatial decay in (Ref. 15, Lemma 6.4), namely* $\psi_0 = \mathcal{O}(|x|^{-2})$ *and* $\psi_1 = \mathcal{O}(|x|^{-1})$ *as* $|x| \to \infty$*. Moreover, it follows from (Ref. 15, Lemma 6.4) that no second generalized eigenvector* $\psi_2 \in H^1(\mathbb{R}, \mathbb{C}^2)$ *exists so that* $\zeta_0 = i$ *is a double eigenvalue of the linear system (1.2) in the sense of Definition 2.1.*

Remark 2.4. The algebraic double-soliton given by (2.20) and (2.21) reduces to the explicit expression obtained in Ref. 8 by using the transformation

$$x \to -x, \quad t \to -t, \quad u \to u, \quad v \to -v,$$

due to a different normalization of the MTM system used in Ref. 8.

III. EXPONENTIAL DOUBLE-SOLITONS FOR A DOUBLE POLE

Here we study solutions of the normalized RH problem for the reflectionless potential $r_\pm(\lambda) \equiv 0$ for $\lambda \in \mathbb{R}$ with a pair of double poles of $M(\lambda)$ at $\lambda_0 \in \mathbb{C}^+$ and $\bar{\lambda}_0 \in \mathbb{C}^-$.

If $m(\lambda)$ is a meromorphic functon near $\lambda_0 \in \mathbb{C}$ with a double pole at λ_0, we represent the principal behavior of $m(\lambda)$ near λ_0 by the Laurent expansion

$$m(\lambda) = \frac{\mathrm{P}^{-2}_{\lambda=\lambda_0} m}{(\lambda-\lambda_0)^2} + \frac{\mathrm{Res}_{\lambda=\lambda_0} m}{\lambda - \lambda_0} + \text{holomorphic at } \lambda = \lambda_0, \tag{3.1}$$

where $\mathrm{P}^{-2}_{\lambda=\lambda_0} m$ and $\mathrm{Res}_{\lambda=\lambda_0} m$ are some coefficients which are referred collectively as the residue coefficients.

The normalized RH problem for $M(\lambda) = (P^\infty)^{-1} P(\lambda)$ can be rewritten in the form:

RH problem. Find a complex-valued analytic function $M(\lambda)$ in $\mathbb{C}\backslash\{\mathbb{R} \cup \{\lambda_0, \bar{\lambda}_0\}\}$ with the following properties:

- $M(\lambda)$ has double poles at $\lambda_0 \in \mathbb{C}^+$ and $\bar{\lambda}_0 \in \mathbb{C}^-$ with the normalization

  $$\mathrm{Res}_{\lambda=\lambda_0} M = (P^\infty)^{-1} \begin{pmatrix} \vec{0} & \mathrm{Res}_{\lambda=\lambda_0} \frac{n_2^{(-)}}{\alpha} \end{pmatrix}, \quad \mathrm{P}^{-2}_{\lambda=\lambda_0} M = (P^\infty)^{-1} \begin{pmatrix} \vec{0} & \mathrm{P}^{-2}_{\lambda=\lambda_0} \frac{n_2^{(-)}}{\alpha} \end{pmatrix},$$

  $$\mathrm{Res}_{\lambda=\bar{\lambda}_0} M = (P^\infty)^{-1} \begin{pmatrix} \mathrm{Res}_{\lambda=\bar{\lambda}_0} \frac{n_1^{(-)}}{\bar{\alpha}} & \vec{0} \end{pmatrix}, \quad \mathrm{P}^{-2}_{\lambda=\bar{\lambda}_0} M = (P^\infty)^{-1} \begin{pmatrix} \mathrm{P}^{-2}_{\lambda=\bar{\lambda}_0} \frac{n_1^{(-)}}{\bar{\alpha}} & \vec{0} \end{pmatrix},$$

  where $\vec{0}$ is a 2-by-1 zero vector.
- $M(\lambda) \to \mathbb{I}$ as $|\lambda| \to \infty$, where $\mathbb{I}$ is the 2-by-2 identity matrix.
- $M(\lambda)$ is continuous on both sides of $\mathbb{R}$ with $M_\pm(\lambda) := \lim_{\mathrm{Im}(\lambda)\to\pm 0} M(\lambda)$ satisfying

  $$M_+(\lambda) = M_-(\lambda), \quad \lambda \in \mathbb{R}.$$

In order to regularize the RH problem, we subtract the residue conditions in both sides of the formula $M_+(\lambda) = M_-(\lambda)$ and obtain the following solution of the normalized RH problem:

$$M(\lambda) = \mathbb{I} + \frac{\mathrm{Res}_{\lambda=\lambda_0} M}{\lambda - \lambda_0} + \frac{\mathrm{Res}_{\lambda=\bar{\lambda}_0} M}{\lambda - \bar{\lambda}_0} + \frac{\mathrm{P}^{-2}_{\lambda=\lambda_0} M}{(\lambda-\lambda_0)^2} + \frac{\mathrm{P}^{-2}_{\lambda=\bar{\lambda}_0} M}{(\lambda-\bar{\lambda}_0)^2}. \tag{3.2}$$

The residue coefficients of $M(\lambda) = (P^\infty)^{-1} P(\lambda)$ near λ_0 and $\bar{\lambda}_0$ in (3.2) follow from the representations (2.11) and (2.12).

A. Computations of the residue coefficients

In order to compute the residue coefficients, we use the following result.

Lemma 3.1. Assume f and g be analytic in a complex region $\Omega \in \mathbb{C}$ such that g has a double zero at $z_0 \in \Omega$ with $g(z_0) = g'(z_0) = 0$, $g''(z_0) \neq 0$, and $f(z_0) \neq 0$. The residue coefficients of the Laurent expansion of f/g at $z = z_0$ are given by

$$\mathrm{Res}_{z=z_0} \frac{f}{g} = \frac{2f'(z_0)}{g''(z_0)} - \frac{2f(z_0)g'''(z_0)}{3[g''(z_0)]^2}, \quad \mathrm{P}^{-2}_{z=z_0} \frac{f}{g} = \frac{2f(z_0)}{g''(z_0)}.$$

Proof. Under conditions of the lemma, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \mathcal{O}((z - z_0)^2),$$
$$g(z) = \frac{1}{2}g''(z_0)(z - z_0)^2 + \frac{1}{6}g'''(z_0)(z - z_0)^3 + \mathcal{O}((z - z_0)^4),$$

from which the result follows by the Laurent expansion of $f(z)/g(z)$. □

Based on Lemma 3.1, we obtain the residue coefficients in the following proposition.

Proposition 3.1. The residue coefficients of $M(\lambda)$ at $\lambda = \lambda_0$ are given by

$$\mathrm{P}^{-2}_{\lambda=\lambda_0} \frac{n_2^{(-)}}{\alpha} = A_0 n_1^{(+)}(\lambda_0) e^{2i\theta(\lambda_0)}, \tag{3.3}$$

$$\mathrm{Res}_{\lambda=\lambda_0} \frac{n_2^{(-)}}{\alpha} = A_0 e^{2i\theta(\lambda_0)} \Big[(n_1^{(+)})'(\lambda_0) + n_1^{(+)}(\lambda_0)\big(2i\theta'(\lambda_0) + B_0\big)\Big], \tag{3.4}$$

where A_0 and B_0 are arbitrary coefficients. The residue conditions of $M(\lambda)$ at $\lambda = \bar{\lambda}_0$ are given by

$$\mathrm{P}^{-2}_{\lambda=\bar{\lambda}_0} \frac{n_1^{(-)}}{\bar{\alpha}} = -\bar{A}_0 \bar{\lambda}_0 n_2^{(+)}(\bar{\lambda}_0) e^{-2i\theta(\bar{\lambda}_0)}, \tag{3.5}$$

$$\mathrm{Res}_{\lambda=\bar{\lambda}_0} \frac{n_1^{(-)}}{\bar{\alpha}} = -\bar{A}_0 \bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)} \Big[(n_2^{(+)})'(\bar{\lambda}_0) + n_2^{(+)}(\bar{\lambda}_0)\big(-2i\theta'(\bar{\lambda}_0) + \bar{B}_0 + \bar{\lambda}_0^{-1}\big)\Big]. \tag{3.6}$$

Proof. By assumption, $\lambda_0 = \zeta_0^2$ is a double zero of $\alpha(\lambda)$ extended to $\mathbb{C}^+$ by Lemma 2.2. Since it folows from (2.2) that

$$\alpha(\lambda) = a(\zeta) = \det(\psi_1^{(+)}(\zeta), \psi_2^{(-)}(\zeta)),$$

we conclude that there exists a constant e_0 such that

$$\psi_2^{(-)}(\zeta_0) = e_0 \psi_1^{(+)}(\zeta_0). \tag{3.7}$$

Furthermore, since ζ_0 is the double zero of $a(\zeta)$, we have

$$\begin{aligned} 0 = a'(\zeta_0) &= \det\Big((\psi_1^{(+)})', \psi_2^{(-)}\Big) + \det\Big(\psi_1^{(+)}, (\psi_2^{(-)})'\Big)\Big|_{\zeta=\zeta_0} \\ &= \det\Big(\psi_1^{(+)}, -e_0(\psi_1^{(+)})' + (\psi_2^{(-)})'\Big)\Big|_{\zeta=\zeta_0}, \end{aligned}$$

so that there exists another constant h_0 such that

$$(\psi_2^{(-)})'(\zeta_0) = e_0(\psi_1^{(+)})'(\zeta_0) + h_0 \psi_1^{(+)}(\zeta_0). \tag{3.8}$$

By using transformation (2.4), we rewrite (3.7) as

$$n_2^{(-)}(\lambda_0) = e_0 \zeta_0^{-1} n_1^{(+)}(\lambda_0) e^{2i\theta(\lambda_0)}, \tag{3.9}$$

where $\theta(\lambda)$ is given by (2.8). This expression agrees with (2.7) for $\alpha(\lambda_0) = 0$. By using transformation (2.4) again and the product rule, we derive

$$\begin{aligned} (n_1^{(+)})'(\lambda_0) = {} & (2\zeta_0)^{-1} T(v, \zeta_0)[(\psi_1^{(+)})'(\zeta_0) - 2i\zeta_0\theta'(\lambda)\psi_1^{(+)}(\zeta_0)]e^{-i\theta(\lambda_0)} \\ & + (2\zeta_0)^{-1}[\partial_\zeta T(v, \zeta_0)]\psi_1^{(+)}(\zeta_0) e^{-i\theta(\lambda_0)}, \end{aligned}$$

$$(n_2^{(-)})'(\lambda_0) = (2\zeta_0^2)^{-1}T(v,\zeta_0)[(\psi_2^{(-)})'(\zeta) + 2i\zeta_0\theta'(\lambda_0)]e^{i\theta(\lambda_0)} - (2\zeta_0^2)^{-1}n_2^{(-)}(\lambda_0) + (2\zeta_0^2)^{-1}[\partial_\zeta T(v,\zeta_0)]\psi_2^{(-)}(\zeta_0)e^{i\theta(\lambda_0)},$$

which imply due to (3.8) and (3.9) that

$$(n_2^{(-)})'(\lambda_0) = e^{2i\theta(\lambda_0)}\Big[e_0\zeta_0^{-1}(n_1^{(+)})'(\lambda_0) + (2\zeta_0^2)^{-1}(h_0 + 4ie_0\zeta_0\theta'(\lambda_0) - e_0\zeta_0^{-1})n_1^{(+)}(\lambda_0)\Big], \tag{3.10}$$

in agreement with the derivative of (2.7) at $\lambda = \lambda_0$.

We use the chain rule

$$\begin{aligned}\alpha'(\lambda) &= (2\zeta)^{-1}a'(\zeta),\\ \alpha''(\lambda) &= (2\zeta)^{-2}[a''(\zeta) - \zeta^{-1}a'(\zeta)],\\ \alpha'''(\lambda) &= (2\zeta)^{-3}[a'''(\zeta) - 3\zeta^{-1}a''(\zeta) + 3\zeta^{-2}a'(\zeta)].\end{aligned}$$

By using (3.9) and (3.10), we compute from the expressions in Lemma 3.1 that

$$\mathrm{P}^{-2}_{\lambda=\lambda_0}\frac{n_2^{(-)}}{\alpha} = \frac{8\zeta_0^2 n_2^{(-)}(\lambda_0)}{a''(\zeta_0)} = \frac{8e_0\zeta_0}{a''(\zeta_0)}\; n_1^{(+)}(\lambda_0)e^{2i\theta(\lambda_0)} \tag{3.11}$$

and

$$\begin{aligned}\mathrm{Res}_{\lambda=\lambda_0}\frac{n_2^{(-)}}{\alpha} &= \frac{8\zeta_0^2(n_2^{(-)})'(\lambda_0)}{a''(\zeta_0)} - \frac{4\zeta_0 n_2^{(-)}(\lambda_0)\big[a'''(\zeta_0) - 3\zeta_0^{-1}a''(\zeta_0)\big]}{3[a''(\zeta_0)]^2},\\ &= \frac{8e_0\zeta_0}{a''(\zeta_0)}e^{2i\theta(\lambda_0)}\left[(n_1^{(+)})'(\lambda_0) + n_1^{(+)}(\lambda_0)\left(2i\theta'(\lambda_0) + \frac{h_0}{2e_0\zeta_0} - \frac{a'''(\zeta_0)}{6\zeta_0 a''(\zeta_0)}\right)\right].\end{aligned} \tag{3.12}$$

Let

$$A_0 = \frac{8e_0\zeta_0}{a''(\zeta_0)}, \qquad B_0 = \frac{h_0}{2e_0\zeta_0} - \frac{a'''(\zeta_0)}{6\zeta_0 a''(\zeta_0)}, \tag{3.13}$$

then (3.11) and (3.12) are transformed into (3.3) and (3.4).

By using the symmetry condition (2.3), we have

$$\psi_1^{(\pm)}(\zeta) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\overline{\psi_2^{(\pm)}}(\zeta), \quad \psi_2^{(\pm)}(\zeta) = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\overline{\psi_1^{(\pm)}}(\zeta),$$

from which we obtain with the help of (3.7) and (3.8) that

$$\psi_1^{(-)}(\bar\zeta_0) = -\bar e_0\psi_2^{(+)}(\bar\zeta_0)$$

and

$$(\psi_1^{(-)})'(\bar\zeta_0) = -\bar e_0(\psi_2^{(+)})'(\bar\zeta_0) - \bar h_0\psi_2^{(+)}(\bar\zeta_0).$$

Furthermore, by using the transformation (2.4) and its derivative, similarly to (3.9) and (3.10), we obtain

$$n_1^{(-)}(\bar\lambda_0) = -\bar e_0\bar\zeta_0 n_2^{(+)}(\bar\lambda_0)e^{-2i\theta(\bar\lambda_0)}$$

and

$$(n_1^{(-)})'(\bar\lambda_0) = -e^{-2i\theta(\bar\lambda_0)}\Big[\bar e_0\bar\zeta_0(n_2^{(+)})'(\bar\lambda_0) + \frac{1}{2}(\bar h_0 - 4i\bar e_0\bar\zeta_0\theta'(\bar\lambda_0) + \bar e_0\bar\zeta_0^{-1})n_2^{(+)}(\bar\lambda_0)\Big].$$

By using these expressions we compute from Lemma 3.1, similarly to (3.11) and (3.12), that

$$\mathrm{P}^{-2}_{\lambda=\bar\lambda_0}\frac{n_1^{(-)}}{\bar\alpha} = -\frac{8\bar e_0\bar\zeta_0^3}{\bar a''(\zeta_0)}\; n_2^{(+)}(\bar\lambda_0)e^{-2i\theta(\bar\lambda_0)} \tag{3.14}$$

and

$$\mathrm{Res}_{\lambda=\bar\lambda_0}\frac{n_1^{(-)}}{\bar\alpha} = -\frac{8\bar e_0\bar\zeta_0^3}{\bar a''(\zeta_0)}e^{-2i\theta(\bar\lambda_0)}\left[(n_2^{(+)})'(\bar\lambda_0) + n_2^{(+)}(\lambda_0)\left(-2i\theta'(\bar\lambda_0) + \frac{\bar h_0}{2\bar e_0\bar\zeta_0} + \frac{1}{\bar\zeta_0^2} - \frac{\bar a'''(\zeta_0)}{6\bar\zeta_0\bar a''(\zeta_0)}\right)\right]. \tag{3.15}$$

Using the same notations (3.13) for A_0 and B_0, we transform (3.14) and (3.15) into (3.5) and (3.6). □

B. Computation of solutions of the linear algebraic system

Using the first column of $M(\lambda)$ in (3.2) for $\lambda\in\mathbb{C}_+$, we obtain from (3.5) and (3.6) that

$$n_1^{(+)}(\lambda)=n_1^{+\infty}e_1-\frac{\bar A_0\bar\lambda_0}{(\lambda-\bar\lambda_0)^2}\big[1+(\lambda-\bar\lambda_0)(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1})\big]n_2^{(+)}(\bar\lambda_0)e^{-2i\theta(\bar\lambda_0)}-\frac{\bar A_0\bar\lambda_0}{\lambda-\bar\lambda_0}(n_2^{(+)})'(\bar\lambda_0)e^{-2i\theta(\bar\lambda_0)}. \tag{3.16}$$

Using the second column of $M(\lambda)$ in (3.2) for $\lambda\in\mathbb{C}_-$, we obtain from (3.3) and (3.4) that

$$n_2^{(+)}(\lambda)=n_2^{+\infty}e_2+\frac{A_0}{(\lambda-\lambda_0)^2}\big[1+(\lambda-\lambda_0)(2i\theta'(\lambda_0)+B_0)\big]n_1^{(+)}(\lambda_0)e^{2i\theta(\lambda_0)}+\frac{A_0}{\lambda-\lambda_0}(n_1^{(+)})'(\lambda_0)e^{2i\theta(\lambda_0)}. \tag{3.17}$$

We can close the algebraic system by evaluating (3.16) at $\lambda=\lambda_0$ and (3.17) at $\lambda=\bar\lambda_0$:

$$\begin{aligned}n_1^{(+)}(\lambda_0)&=n_1^{+\infty}e_1-\bar C_0\bar\lambda_0\big[1+(\lambda_0-\bar\lambda_0)(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1})\big]n_2^{(+)}(\bar\lambda_0)e^{-2i\theta(\bar\lambda_0)}-\bar C_0\bar\lambda_0(\lambda_0-\bar\lambda_0)(n_2^{(+)})'(\bar\lambda_0)e^{-2i\theta(\bar\lambda_0)},\\ n_2^{(+)}(\bar\lambda_0)&=n_2^{+\infty}e_2+C_0\big[1-(\lambda_0-\bar\lambda_0)(2i\theta'(\lambda_0)+B_0)\big]n_1^{(+)}(\lambda_0)e^{2i\theta(\lambda_0)}-C_0(\lambda_0-\bar\lambda_0)(n_1^{(+)})'(\lambda_0)e^{2i\theta(\lambda_0)},\end{aligned} \tag{3.18}$$

as well as their derivatives at $\lambda=\lambda_0$ and $\lambda=\bar\lambda_0$ respectively:

$$\begin{aligned}(n_1^{(+)})'(\lambda_0)&=\bar C_0\bar\lambda_0\big[2(\lambda_0-\bar\lambda_0)^{-1}-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1}\big]n_2^{(+)}(\bar\lambda_0)e^{-2i\theta(\bar\lambda_0)}+\bar C_0\bar\lambda_0(n_2^{(+)})'(\bar\lambda_0)e^{-2i\theta(\bar\lambda_0)},\\ (n_2^{(+)})'(\bar\lambda_0)&=C_0\big[2(\lambda_0-\bar\lambda_0)^{-1}-2i\theta'(\lambda_0)-B_0\big]n_1^{(+)}(\lambda_0)e^{2i\theta(\lambda_0)}-C_0(n_1^{(+)})'(\lambda_0)e^{2i\theta(\lambda_0)},\end{aligned} \tag{3.19}$$

where

$$C_0:=\frac{A_0}{(\lambda_0-\bar\lambda_0)^2}.$$

The following proposition solves the linear system (3.18) and (3.19) and derives the explicit representation for the exponential double-soliton solution of the MTM system (1.1) by using the recovery formulas (2.13).

Proposition 3.2. The potentials $u(x,t)$ and $v(x,t)$ in (2.13) are expressed from solutions of the RH problem with a double pole by

$$u=\frac{\overline{N_u}}{\overline{D}},\quad v=\frac{N_v}{D}, \tag{3.20}$$

where

$$\begin{aligned}N_u=&-\lambda_0^{-1}A_0e^{2i\theta(\lambda_0)}(2i\theta'(\lambda_0)+B_0-\lambda_0^{-1})-\lambda_0^{-1}A_0|C_0|^2\bar\lambda_0e^{4i\theta(\lambda_0)-2i\theta(\bar\lambda_0)}\\ &\times[4(\lambda_0-\bar\lambda_0)^{-1}+(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1})\bar\lambda_0\lambda_0^{-1}-3\lambda_0^{-1}],\end{aligned} \tag{3.21}$$

$$\begin{aligned}N_v=&\bar A_0e^{-2i\theta(\bar\lambda_0)}(-2i\theta'(\bar\lambda_0)+\bar B_0)+\bar A_0|C_0|^2\bar\lambda_0e^{2i\theta(\lambda_0)-4i\theta(\bar\lambda_0)}\\ &\times[-4(\lambda_0-\bar\lambda_0)^{-1}+(2i\theta'(\lambda_0)+B_0)\lambda_0\bar\lambda_0^{-1}-3\bar\lambda_0^{-1}].\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}D=&1+|C_0|^4\bar\lambda_0^2e^{4i\theta(\lambda_0)-4i\theta(\bar\lambda_0)}+|C_0|^2\bar\lambda_0e^{2i\theta(\lambda_0)-2i\theta(\bar\lambda_0)}\big[6+2(\lambda_0-\bar\lambda_0)(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1}-2i\theta'(\lambda_0)-B_0)\\ &-(\lambda_0-\bar\lambda_0)^2(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1})(2i\theta'(\lambda_0)+B_0)\big]\end{aligned} \tag{3.23}$$

Proof. By substituting (3.2), (3.3), and (3.4) into (2.13), we obtain

$$\begin{aligned}\overline{u}&=\lim_{\lambda\to0}M_{12}(\lambda)\\ &=-\frac{1}{\lambda_0}n_2^{+\infty}\mathrm{Res}_{\lambda=\lambda_0}\left[\frac{n_{21}^{(-)}(\lambda)}{\alpha(\lambda)}\right]+\frac{1}{\lambda_0^2}n_2^{+\infty}\mathrm{P}^{-2}_{\lambda=\lambda_0}\left[\frac{n_{21}^{(-)}(\lambda)}{\alpha(\lambda)}\right],\\ &=-\frac{1}{\lambda_0}n_2^{+\infty}A_0e^{2i\theta(\lambda_0)}\Big[(n_{11}^{(+)})'(\lambda_0)+n_{11}^{(+)}(\lambda_0)(2i\theta'(\lambda_0)+B_0-\lambda_0^{-1})\Big],\end{aligned} \tag{3.24}$$

where the second index for vectors $n_1^{(+)}$ and $n_2^{(-)}$ denotes the corresponding components of 2-vectors. Similarly, by substituting (3.2), (3.5), and (3.6) into (2.13), we obtain

$$\begin{aligned} v &= \lim_{\lambda\to 0} M_{21}(\lambda) \\ &= -\frac{1}{\bar\lambda_0} n_1^{+\infty}\operatorname{Res}_{\lambda=\bar\lambda_0}\left[\frac{n_{12}^{(-)}(\lambda)}{\bar\alpha(\lambda)}\right] + \frac{1}{\bar\lambda_0^2} n_1^{+\infty}\mathrm{P}^{-2}_{\lambda=\bar\lambda_0}\left[\frac{n_{12}^{(-)}(\lambda)}{\bar\alpha(\lambda)}\right], \\ &= n_1^{+\infty}\bar A_0 e^{-2i\theta(\bar\lambda_0)}\left[(n_{22}^{(+)})'(\bar\lambda_0) + n_{22}^{(+)}(\bar\lambda_0)(-2i\theta'(\bar\lambda_0)+\bar B_0)\right]. \end{aligned} \tag{3.25}$$

The linear system (3.18) and (3.19) can be rewritten for the vectors $n_1^{(+)}(\lambda_0)$ and $(n_1^{(+)})'(\lambda_0)$:

$$K\begin{pmatrix} n_1^{(+)}(\lambda_0) \\ (n_1^{(+)})'(\lambda_0)\end{pmatrix} = \begin{pmatrix} n_1^{+\infty}e_1 - \bar C_0\bar\lambda_0\left[1+(\lambda_0-\bar\lambda_0)(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1})\right]n_2^{+\infty}e^{-2i\theta(\bar\lambda_0)}e_2 \\ \bar C_0\bar\lambda_0\left[2(\lambda_0-\bar\lambda_0)^{-1} - 2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1}\right]n_2^{+\infty}e^{-2i\theta(\bar\lambda_0)}e_2\end{pmatrix},$$

where

$$K = \begin{pmatrix} K_{11}\mathbb{I} & K_{12}\mathbb{I} \\ K_{21}\mathbb{I} & K_{22}\mathbb{I}\end{pmatrix}$$

with $\mathbb{I}$ being a 2-by-2 identity matrix and M_{ij} being scalar entries given by

$$\begin{aligned} K_{11} &= 1 + |C_0|^2\bar\lambda_0 e^{2i\theta(\lambda_0)-2i\theta(\bar\lambda_0)}\left[3+(\lambda_0-\bar\lambda_0)(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1}) - 2(\lambda_0-\bar\lambda_0)(2i\theta'(\lambda_0)+B_0)\right. \\ &\quad \left. - (\lambda_0-\bar\lambda_0)^2(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1})(2i\theta'(\lambda_0)+B_0)\right], \\ K_{12} &= -|C_0|^2\bar\lambda_0 e^{2i\theta(\lambda_0)-2i\theta(\bar\lambda_0)}(\lambda_0-\bar\lambda_0)\left[2+(\lambda_0-\bar\lambda_0)(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1})\right], \\ K_{21} &= -|C_0|^2\bar\lambda_0(\lambda_0-\bar\lambda_0)^{-1}e^{2i\theta(\lambda_0)-2i\theta(\bar\lambda_0)}\left[4+(\lambda_0-\bar\lambda_0)(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1})\right. \\ &\quad \left. - 3(\lambda_0-\bar\lambda_0)(2i\theta'(\lambda_0)+B_0) - (\lambda_0-\bar\lambda_0)^2(2i\theta'(\lambda_0)+B_0)(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1})\right], \\ K_{22} &= 1+|C_0|^2\bar\lambda_0 e^{2i\theta(\lambda_0)-2i\theta(\bar\lambda_0)}\left[3+(\lambda_0-\bar\lambda_0)(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1})\right]. \end{aligned}$$

By Cramer's rule, we obtain the first components of vectors $n_1^{(+)}(\lambda_0)$ and $(n_1^{(+)})'(\lambda_0)$:

$$n_{11}^{(+)}(\lambda_0) = \frac{n_1^\infty K_{22}}{D}, \qquad (n_{11}^{(+)})'(\lambda_0) = \frac{-n_1^\infty K_{21}}{D}, \tag{3.26}$$

where $D = K_{11}K_{22} - K_{12}K_{21}$ recovers (3.23) after cancellation of several terms proportional to $|C_0|^4$. Substituting (3.26) into (3.24), we get u in the form (3.20) with

$$N_u = -\lambda_0^{-1}A_0 e^{2i\theta(\lambda_0)}\left[-K_{21}+K_{22}(2i\theta'(\lambda_0)+B_0-\lambda_0^{-1})\right]$$

which yields (3.21) after cancellation of several terms proportional to $|C_0|^2$.

For the vectors $n_2^{(+)}(\lambda_0)$ and $(n_2^{(+)})'(\lambda_0)$, the linear system (3.18) and (3.19) can be rewritten as

$$\tilde K\begin{pmatrix} n_2^{(+)}(\lambda_0) \\ (n_2^{(+)})'(\lambda_0)\end{pmatrix} = \begin{pmatrix} n_2^{+\infty}e_2 + C_0\left[1-(\lambda_0-\bar\lambda_0)(2i\theta'(\lambda_0)+B_0)\right]n_1^{+\infty}e^{2i\theta(\lambda_0)}e_1 \\ C_0\left[2(\lambda_0-\bar\lambda_0)^{-1} - 2i\theta'(\lambda_0) - B_0\right]n_1^{+\infty}e^{2i\theta(\lambda_0)}e_1\end{pmatrix},$$

where

$$\tilde K = \begin{pmatrix} \tilde K_{11}\mathbb{I} & \tilde K_{12}\mathbb{I} \\ \tilde K_{21}\mathbb{I} & \tilde K_{22}\mathbb{I}\end{pmatrix}$$

with

$$\begin{aligned} \tilde K_{11} &= 1+|C_0|^2\bar\lambda_0 e^{2i\theta(\lambda_0)-2i\theta(\bar\lambda_0)}\left[3+2(\lambda_0-\bar\lambda_0)(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1}) - (\lambda_0-\bar\lambda_0)(2i\theta'(\lambda_0)+B_0)\right. \\ &\quad \left. - (\lambda_0-\bar\lambda_0)^2(-2i\theta'(\bar\lambda_0)+\bar B_0+\bar\lambda_0^{-1})(2i\theta'(\lambda_0)+B_0)\right], \\ \tilde K_{12} &= |C_0|^2\bar\lambda_0 e^{2i\theta(\lambda_0)-2i\theta(\bar\lambda_0)}(\lambda_0-\bar\lambda_0)\left[2-(\lambda_0-\bar\lambda_0)(2i\theta'(\lambda_0)+B_0)\right], \end{aligned}$$

$$\tilde{K}_{21} = |C_0|^2\bar{\lambda}_0(\lambda_0 - \bar{\lambda}_0)^{-1}e^{2i\theta(\lambda_0)-2i\theta(\bar{\lambda}_0)}\big[4 + 3(\lambda_0 - \bar{\lambda}_0)(-2i\theta'(\bar{\lambda}_0) + \bar{B}_0 + \bar{\lambda}_0^{-1})$$
$$- (\lambda_0 - \bar{\lambda}_0)(2i\theta'(\lambda_0) + B_0) - (\lambda_0 - \bar{\lambda}_0)^2(2i\theta'(\lambda_0) + B_0)(-2i\theta'(\bar{\lambda}_0) + \bar{B}_0 + \bar{\lambda}_0^{-1})\big],$$
$$\tilde{K}_{22} = 1 + |C_0|^2\bar{\lambda}_0 e^{2i\theta(\lambda_0)-2i\theta(\bar{\lambda}_0)}[3 - (\lambda_0 - \bar{\lambda}_0)(2i\theta'(\lambda_0) + B_0)].$$

By Cramer's rule, we obtain the second components of vectors $n_2^{(+)}(\lambda_0)$ and $(n_2^{(+)})'(\lambda_0)$:

$$n_{22}^{(+)}(\lambda_0) = \frac{n_2^{\infty}\tilde{K}_{22}}{\tilde{D}}, \quad (n_{22}^{(+)})'(\lambda_0) = \frac{-n_2^{\infty}\tilde{K}_{21}}{\tilde{D}}, \tag{3.27}$$

where $\tilde{D} = \tilde{K}_{11}\tilde{K}_{22} - \tilde{K}_{12}\tilde{K}_{21} = D$ is given by (3.23). Substituting (3.27) into (3.25), we get v in the form (3.20) with

$$N_v = \bar{A}_0 e^{-2i\theta(\bar{\lambda}_0)}\big[-\tilde{K}_{21} + \tilde{K}_{22}(-2i\theta'(\bar{\lambda}_0) + \bar{B}_0)\big],$$

which yields (3.22) after cancellation of several terms proportional to $|C_0|^2$. □

C. Proof of theorem 2.1

In order to rewrite the recovered potentials of Proposition 3.2 in the simplified form of Theorem 2.1, we set $\lambda_0 = e^{i\gamma} \in \mathbb{S}^1 \cap \mathbb{C}^+$ with $\gamma \in (0, \pi)$. A more general solution is obtained with the Lorentz symmetry (2.14).

Since $\lambda_0 = e^{i\gamma}$, we obtain

$$2i\theta(\lambda_0) = -x\sin\gamma + it\cos\gamma,$$
$$4i\theta'(\lambda_0) = i(x+t) + i(x-t)e^{-2i\gamma},$$

and complex conjugate for $-2i\theta(\bar{\lambda}_0)$ and $-4i\theta'(\bar{\lambda}_0)$.

Let us define $A_0 = (\lambda_0 - \bar{\lambda}_0)^2\lambda_0^{3/2} = -4(\sin\gamma)^2 e^{\frac{3i\gamma}{2}}$, which yields $C_0 = e^{\frac{3i\gamma}{2}}$. A more general solution with two translational parameters $x_0, t_0 \in \mathbb{R}$ can be obtained by the translational symmetry (2.15) or by including two additional parameters in $A_0 \in \mathbb{C}$. Then it follows from (3.21), (3.22), and (3.23) that

$$N_u = 2i(\sin\gamma)^2 e^{-x\sin\gamma + it\cos\gamma + \frac{i\gamma}{2}}\big(x + t + (x-t)e^{-2i\gamma} - 2iB_0 + 2ie^{-i\gamma}$$
$$- e^{-2x\sin\gamma - i\gamma}[4(\sin\gamma)^{-1} + (x + t + (x-t)e^{2i\gamma} + 2i\bar{B}_0 + 2ie^{i\gamma})e^{-2i\gamma} - 6ie^{-i\gamma}]\big),$$
$$N_v = 2i(\sin\gamma)^2 e^{-x\sin\gamma - it\cos\gamma - \frac{3i\gamma}{2}}\big(x + t + (x-t)e^{2i\gamma} + 2i\bar{B}_0$$
$$- e^{-2x\sin\gamma - i\gamma}[4(\sin\gamma)^{-1} + (x + t + (x-t)e^{-2i\gamma} - 2iB_0)e^{2i\gamma} + 6ie^{i\gamma}]\big),$$

and

$$D = 1 + e^{-4x\sin\gamma - 2i\gamma} + e^{-2x\sin\gamma - i\gamma}\big[6 + 4(\sin\gamma)(x + t + (x-t)\cos 2\gamma + i(\bar{B}_0 - B_0 + e^{i\gamma}))$$
$$+ (\sin\gamma)^2(x + t + (x-t)e^{2i\gamma} + 2i\bar{B}_0 + 2ie^{i\gamma})(x + t + (x-t)e^{-2i\gamma} - 2iB_0)\big].$$

By using trigonometric identities, we reduce expressions for N_u, N_v and D to the form:

$$N_u = 4i(\sin\gamma)^2 e^{-x\sin\gamma + it\cos\gamma - \frac{i\gamma}{2}}\big(x\cos\gamma + it\sin\gamma - iB_0e^{i\gamma} + i - e^{-2x\sin\gamma - i\gamma}[2\cot\gamma + x\cos\gamma - it\sin\gamma + i\bar{B}_0e^{-i\gamma}]\big),$$
$$N_v = 4i(\sin\gamma)^2 e^{-x\sin\gamma - it\cos\gamma - \frac{i\gamma}{2}}\big(x\cos\gamma - it\sin\gamma + i\bar{B}_0e^{-i\gamma} - e^{-2x\sin\gamma - i\gamma}[2\cot\gamma + x\cos\gamma + it\sin\gamma - iB_0e^{i\gamma} + i]\big),$$
$$D = 1 + e^{-4x\sin\gamma - 2i\gamma} + 2e^{-2x\sin\gamma - i\gamma}\big[3 + 2(\sin\gamma)(2x(\cos\gamma)^2 + 2t(\sin\gamma)^2 + i(\bar{B}_0 - B_0 + e^{i\gamma}))$$
$$+ 2(\sin\gamma)^2(x\cos\gamma - it\sin\gamma + i\bar{B}_0e^{-i\gamma} + i)(x\cos\gamma + it\sin\gamma - iB_0e^{i\gamma})\big].$$

By selecting $B_0 = -ie^{-i\gamma}[\tilde{x}_0\cos\gamma + i\tilde{t}_0\sin\gamma]$ with arbitrary parameters $\tilde{x}_0$ and $\tilde{t}_0$, we obtain the same expressions for N_u and N_v as in Theorem 2.1. Regarding the expression for D, we obtain

$$D = 1 + e^{-4x\sin\gamma - 2i\gamma} + 2e^{-2x\sin\gamma - i\gamma}\big[3 + 2(\sin\gamma)[2(x - \tilde{x}_0)(\cos\gamma)^2 + 2(t - \tilde{t}_0)(\sin\gamma)^2 + i\cos\gamma - \sin\gamma]$$
$$+ 2(\sin\gamma)^2[(x - \tilde{x}_0)\cos\gamma - i(t - \tilde{t}_0)\sin\gamma + i][(x - \tilde{x}_0)\cos\gamma + i(t - \tilde{t}_0)\sin\gamma)]\big].$$

Expanding the bracket multiplied by $2e^{-2x\sin\gamma - i\gamma}$ yields

$$\begin{aligned}&3-2(\sin\gamma)^2+2i(\sin\gamma)(\cos\gamma)+4(\sin\gamma)(\cos\gamma)^2(x-\tilde{x}_0)+2(\sin\gamma)^3(t-\tilde{t}_0)\\&\quad+2(\sin\gamma)^2(\cos\gamma)^2(x-\tilde{x}_0)^2+2i(\sin\gamma)^2(\cos\gamma)(x-\tilde{x}_0)+2(\sin\gamma)^4(t-\tilde{t}_0)^2\\&=1+2(\sin\gamma)^2\left[\cot\gamma+(x-\tilde{x}_0)\cos\gamma+\frac{i}{2}\right]^2+2(\sin\gamma)^4\left[t-\tilde{t}_0+\frac{1}{2\sin\gamma}\right]^2,\end{aligned}$$

which coincides with the expression for D in Theorem 2.1.

D. Computations of eigenvectors and generalized eigenvectors

An eigenvector of the linear system (2.18) is given by $\psi_1^{(+)}(\zeta_0)$ for $\zeta_0 = e^{\frac{i\gamma}{2}}$ with $\gamma \in (0,\pi)$, which decays exponentially as $|x| \to \infty$ due to (2.1) and (3.7). By using the transformation (2.4), we obtain

$$\psi_1^{(+)}(\zeta_0) = e^{i\theta(\lambda_0)}[T(v,\zeta_0)]^{-1} n_1^{(+)}(\lambda_0), \tag{3.28}$$

where

$$[T(v,\zeta_0)]^{-1} = \begin{pmatrix} 1 & 0 \\ -e^{-\frac{i\gamma}{2}} v & e^{-\frac{i\gamma}{2}} \end{pmatrix}$$

and $n_1^{(+)}(\lambda_0)$ with $\lambda_0 = e^{i\gamma}$ is obtained from the linear system in the proof of Proposition 3.2. By Cramer's rule, as in (3.26), we obtain

$$n_1^{(+)}(\lambda_0) = \frac{1}{D} P^\infty \begin{pmatrix} K_{22} \\ -\bar{C}_0 \bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)}(K_{22} b_1 + K_{12} b_2) \end{pmatrix}, \tag{3.29}$$

where

$$\begin{aligned} b_1 &= 1 + (\lambda_0 - \bar{\lambda}_0)(-2i\theta'(\bar{\lambda}_0) + \bar{B}_0 + \bar{\lambda}_0^{-1}), \\ b_2 &= (\lambda_0 - \bar{\lambda}_0)^{-1}(2 + (\lambda_0 - \bar{\lambda}_0)(-2i\theta'(\bar{\lambda}_0) + \bar{B}_0 + \bar{\lambda}_0^{-1})), \end{aligned}$$

and we recall that $P^\infty = \mathrm{diag}\,(n_1^{+\infty}, n_2^{+\infty})$ with

$$n_1^{+\infty} = e^{\frac{i}{4}\int_{+\infty}^{x}(|u|^2+|v|^2)dy} = \bar{n}_2^{+\infty}.$$

By using the same definitions of A_0 and B_0 as in the proof of Theorem 2.1, we obtain

$$\begin{aligned} K_{12} &= -4i(\sin\gamma)^2 e^{-2x\sin\gamma}(\cot\gamma + (x-\tilde{x}_0)\cos\gamma - i(t-\tilde{t}_0)\sin\gamma), \\ K_{22} &= 1 + (\sin\gamma)e^{-2x\sin\gamma}(3\cot\gamma + 2(x-\tilde{x}_0)\cos\gamma - 2i(t-\tilde{t}_0)\sin\gamma - i) \end{aligned}$$

which yields the second component of the vector $n_1^{(+)}(\lambda_0)$ in the explicit form:

$$\begin{aligned} -\bar{C}_0\bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)}(K_{22}b_1 + K_{12}b_2) &= -e^{-x\sin\gamma - it\cos\gamma - \frac{3i\gamma}{2}} \\ &\quad \times \left(e^{i\gamma} + 2\sin\gamma[(x-\tilde{x}_0)\cos\gamma - i(t-\tilde{t}_0)\sin\gamma] - e^{-2x\sin\gamma - 2i\gamma}\right). \end{aligned}$$

Solutions of the Lax pair of linear Eq. (1.2) depend analytically on the spectral parameter ζ near $\zeta_0 = e^{\frac{i\gamma}{2}}$. As a result, derivative of the solution in ζ at $\zeta = \zeta_0$ satisfies the system (2.19) for the generalized eigenvector and generates the generalized eigenvector provided that the derivative of the solution decays exponentially as $|x| \to \infty$. Hence, we define here $(\psi_1^{(+)})'(\zeta_0)$ and confirm the exponential decay as $|x| \to \infty$ which follows from (2.1), (3.7), and (3.8).

By differentiating the transformation (2.4) in $\lambda = \zeta^2$ at $\lambda_0 = \zeta_0^2 = e^{i\gamma}$, we obtain

$$(\psi_1^{(+)})'(\zeta_0) = 2\zeta_0 e^{i\theta(\lambda_0)}[T(v,\zeta_0)]^{-1}(n_1^{(+)})'(\lambda_0) + 2i\zeta_0\theta'(\lambda_0)\psi_1^{(+)}(\zeta_0) - \zeta_0^{-1}\partial_\zeta T(v,\zeta)\psi_1^{(+)}(\zeta_0), \tag{3.30}$$

where $\psi_1^{(+)}(\zeta_0)$ is given by the exponentially decaying eigenfunction (3.28) and

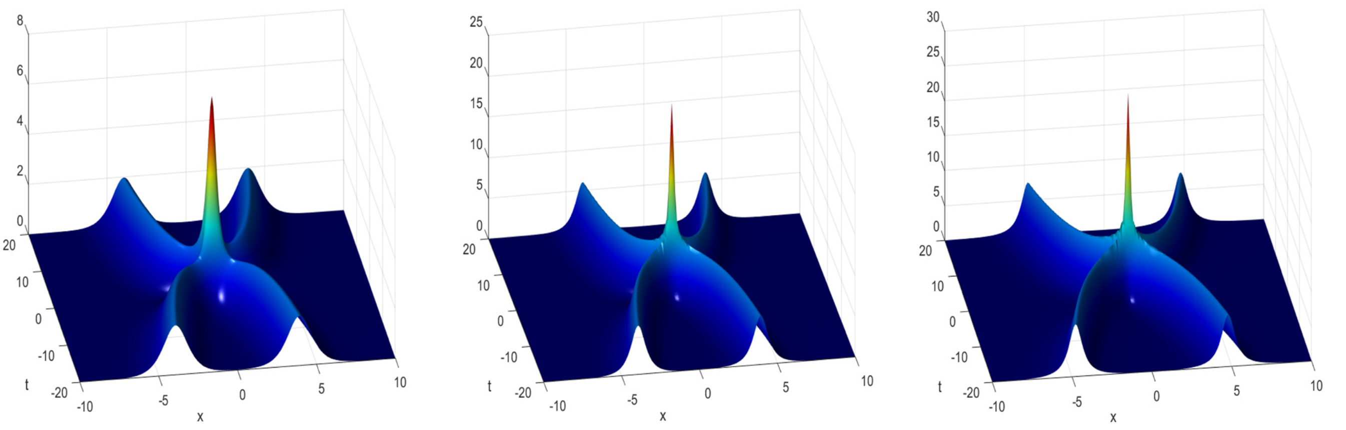

FIG. 1. The surface plots of $|u(x,t)|^2 + |v(x,t)|^2$ for the exponential double-soliton solutions with (a) $\gamma = \frac{\pi}{3}$, (b) $\gamma = \frac{2\pi}{3}$, and (c) $\gamma = \frac{5\pi}{6}$.

$$\partial_\zeta T(v,\zeta) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, in order to obtain $(\psi_1^{(+)})'(\zeta_0)$, we only need to compute $(n_1^{(+)})'(\lambda_0)$ and use the transformation (3.30). By Cramer's rule, as in (3.26), we obtain

$$(n_1^{(+)})'(\lambda_0) = \frac{1}{D} P^\infty \begin{pmatrix} -K_{21} \\ \bar{C}_0 \bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)} (K_{11} b_2 + K_{21} b_1) \end{pmatrix}. \tag{3.31}$$

Proceeding similarly, we obtain

$$\begin{aligned} K_{11} &= 1 + e^{-2x \sin\gamma - i\gamma} \big(3 + 2(\sin\gamma) e^{i\gamma} [(x - \tilde{x}_0)\cos\gamma - i(t - \tilde{t}_0)\sin\gamma + i] \\ &\quad + 4(\sin\gamma) e^{-i\gamma} [(x - \tilde{x}_0)\cos\gamma + i(t - \tilde{t}_0)\sin\gamma] + 4(\sin\gamma)^2 [(x - \tilde{x}_0)\cos\gamma - i(t - \tilde{t}_0)\sin\gamma + i][(x - \tilde{x}_0)\cos\gamma + i(t - \tilde{t}_0)\sin\gamma]\big), \\ K_{21} &= \frac{i}{\sin\gamma} e^{-2x \sin\gamma - i\gamma} \big(2 + (\sin\gamma) e^{i\gamma} [(x - \tilde{x}_0)\cos\gamma - i(t - \tilde{t}_0)\sin\gamma + i] \\ &\quad + 3(\sin\gamma) e^{-i\gamma} [(x - \tilde{x}_0)\cos\gamma + i(t - \tilde{t}_0)\sin\gamma] + 2(\sin\gamma)^2 [(x - \tilde{x}_0)\cos\gamma - i(t - \tilde{t}_0)\sin\gamma + i][(x - \tilde{x}_0)\cos\gamma + i(t - \tilde{t}_0)\sin\gamma]\big), \end{aligned}$$

which yields the second component of the vector $(n_1^{(+)})'(\lambda_0)$ in the explicit form:

$$\begin{aligned} \bar{C}_0 \bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)} (K_{11} b_2 + K_{21} b_1) &= -ie^{-x \sin\gamma - it \cos\gamma - \frac{3i}{2}\gamma} \big(\cot\gamma + (x - \tilde{x}_0)\cos\gamma - i(t - \tilde{t}_0)\sin\gamma \\ &\quad + e^{-2x \sin\gamma - 3i\gamma} [\cot\gamma + (x - \tilde{x}_0)\cos\gamma + i(t - \tilde{t}_0)\sin\gamma + i]\big). \end{aligned}$$

Remark 3.1. The explicit expressions (3.29) and (3.31) confirm that both the eigenvector (3.28) and the generalized eigenvector (3.30) decay exponentially as $x \to \pm\infty$ since $D \to 1$ as $x \to +\infty$ and $D \sim e^{-4x\sin\gamma - 2i\gamma}$ as $x \to -\infty$.

E. Numerical illustration of the exponential double-solitons

We plot the exponential double-soliton solutions of Theorem 2.1 in Fig. 1 for three different values of γ. The translational parameters in (2.17) are set to $\tilde{x}_0 = \frac{1}{\sin\gamma}$ and $\tilde{t}_0 = \frac{1}{2\sin\gamma}$. The solutions describe scattering of two identical solitons which slowly approach to each other, overlap, and then slowly diverge from each other. Similar solutions for the cubic NLS equation were constructed for the first time in Ref. 36.

We shall find the approximate distance between the two identical solitons for large $|x| + |t|$. It follows from the bilinear equations, see Refs. 4 and 8 and Appendix B, that

$$|u|^2 + |v|^2 = \frac{|N_u|^2 + |N_v|^2}{|D|^2} = 2i \frac{\partial}{\partial x} \log \frac{\overline{D}}{D}. \tag{3.32}$$

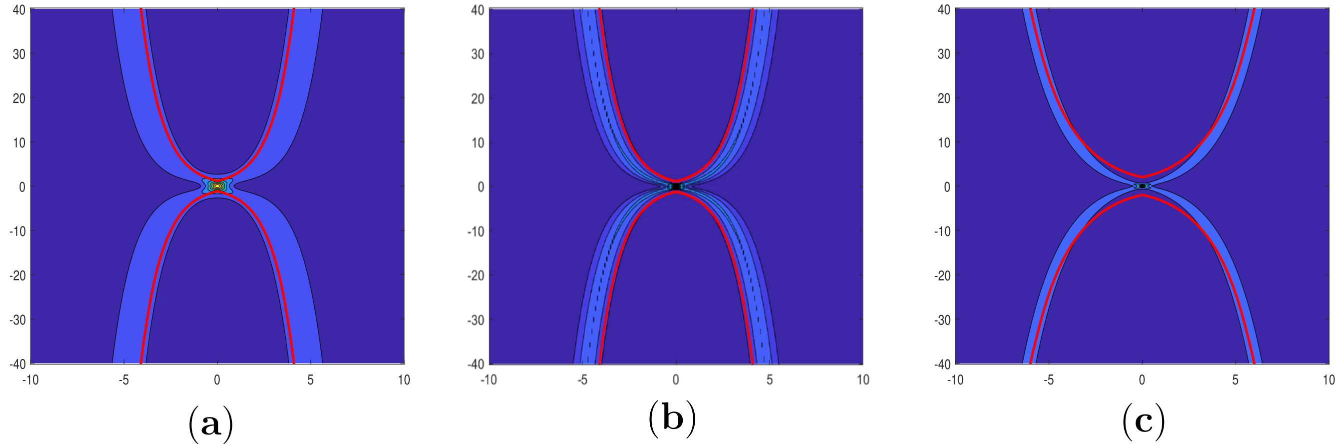

FIG. 2. The contour plots of $|u(x,t)|^2 + |v(x,t)|^2$ for the solutions of Fig. 1 with (a) $\gamma = \frac{\pi}{3}$, (b) $\gamma = \frac{2\pi}{3}$, (c) $\gamma = \frac{5\pi}{6}$. The red line represents the dependence (3.33) for $|t| \geq 1$.

Therefore, we just need to investigate the behavior of D for large $|x| + |t|$. The dominant terms of D as $|x| + |t| \to \infty$ are given by

$$D \sim e^{-2x \sin\gamma - i\gamma}\left(e^{-2x\sin\gamma - i\gamma} + 4(\sin\gamma)^4 t^2\right),$$

from which we obtain that

$$|x| \sim \frac{\ln|t|}{\sin\gamma}, \quad \text{as} \quad |x| + |t| \to \infty. \tag{3.33}$$

The dependence (3.33) for $|t| \geq 1$ is shown in Fig. 2 by red line together with the contour plots from Fig. 1.

IV. LIMIT TO THE ALGEBRAIC DOUBLE-SOLITONS

Here we take the limit $\gamma \to \pi$ of the exponential double-solitons in Theorem 2.1 to derive the algebraic double-solitons. We show that the algebraic double-solitons correspond to the double embedded eigenvalue $\zeta_0 = i$ in the linear systems (2.18) and (2.19). In order to obtain nontrivial limits, we change the arbitrary parameters $\tilde{x}_0$ and $\tilde{t}_0$ used in Sec. III as

$$\tilde{x}_0 = \hat{x}_0 + \frac{1}{\sin\gamma}, \qquad \tilde{t}_0 = \hat{t}_0 + \frac{1}{2\sin\gamma}. \tag{4.1}$$

The two computations below give the proof of Theorem 2.2.

A. Computations of (2.20) and (2.21)

Let $\gamma := \pi - \varepsilon$ and consider the limit $\varepsilon \to 0^+$. Taylor's expansions yield

$$\sin\gamma = \varepsilon - \frac{1}{6}\varepsilon^3 + \mathcal{O}(\varepsilon^5),$$
$$\cos\gamma = -1 + \frac{\varepsilon^2}{2} + \mathcal{O}(\varepsilon^4).$$

To obtain (2.20) and (2.21), we only need to substitute (4.1) into D, N_u and N_v given below (2.17) and collect together the coefficients of Taylor expansion at powers ε, ε^2, ε^3, and ε^4. With the substitution (4.1), we rewrite D as

$$D = 1 + e^{-4x\sin\gamma - 2i\gamma} + 2e^{-2x\sin\gamma - i\gamma}\left(1 + 2(\sin\gamma)^2\left[(x - \hat{x}_0)\cos\gamma + \frac{i}{2}\right]^2 + 2(\sin\gamma)^4(t - \hat{t}_0)^2\right).$$

Expansion as $\varepsilon \to 0$ gives nonzero terms at powers ε^2, ε^3, and ε^4:

$$\varepsilon^2: \qquad -4\hat{x}_0(i - 2x + \hat{x}_0),$$
$$\varepsilon^3: \qquad -4\hat{x}_0(i - 2x)(i - 2x + \hat{x}_0),$$

$$\varepsilon^4: \quad \frac{1}{12}(i-2x)^4 + \frac{1}{3}(i-2x)(i-6x) - 4(t-\hat{t}_0)^2 - 2\hat{x}_0(i-2x)^2(i-2x+\hat{x}_0) + \frac{2}{3}\hat{x}_0(8\hat{x}_0 - 16x + 5i).$$

With the transformation (4.1), we rewrite N_u and N_v as

$$N_u = 4i(\sin\gamma)^2 e^{-x\sin\gamma + it\cos\gamma - \frac{i\gamma}{2}}\Big(-\cot\gamma + (x-\hat{x}_0)\cos\gamma + i(t-\hat{t}_0)\sin\gamma + \frac{i}{2} - e^{-2x\sin\gamma - i\gamma}\Big[\cot\gamma + (x-\hat{x}_0)\cos\gamma - i(t-\hat{t}_0)\sin\gamma + \frac{i}{2}\Big]\Big)$$

and

$$N_v = 4i(\sin\gamma)^2 e^{-x\sin\gamma - it\cos\gamma - \frac{i\gamma}{2}}\Big(-\cot\gamma + (x-\hat{x}_0)\cos\gamma - i(t-\hat{t}_0)\sin\gamma + \frac{i}{2} - e^{-2x\sin\gamma - i\gamma}\Big[\cot\gamma + (x-\hat{x}_0)\cos\gamma + i(t-\hat{t}_0)\sin\gamma + \frac{i}{2}\Big]\Big).$$

The expressions in the parentheses for N_u and N_v are multiplied by

$$4i(\sin\gamma)^2 e^{-x\sin\gamma \pm it\cos\gamma - \frac{i\gamma}{2}} \sim 4\varepsilon^2 e^{\mp it} \quad \text{as} \quad \varepsilon \to 0.$$

Since the expansions of the exponential factors in powers of ε do not modify the limit of $\varepsilon \to 0$, we collect nonzero terms in the expansions of $N_u e^{-it\cos\gamma}$ and $N_v e^{it\cos\gamma}$ at powers ε^2, ε^3, and ε^4:

$$\begin{aligned}
\varepsilon^2: &\quad 8\hat{x}_0,\\
\varepsilon^3: &\quad 4\hat{x}_0(i-2x),\\
\varepsilon^4: &\quad \frac{1}{3}(i-2x)^3 + 5\hat{x}_0(i-2x)^2 - 4i(t-\hat{t}_0)(i-2x) + \frac{4i}{3} - \frac{20}{3}\hat{x}_0,
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon^2: &\quad 8\hat{x}_0,\\
\varepsilon^3: &\quad 4\hat{x}_0(i-2x),\\
\varepsilon^4: &\quad \frac{1}{3}(i-2x)^3 + 5\hat{x}_0(i-2x)^2 + 4i(t-\hat{t}_0)(i-2x) + \frac{4i}{3} - \frac{20}{3}\hat{x}_0.
\end{aligned}$$

By rescaling $\hat{x}_0 = \check{x}_0\varepsilon^2$, we now obtain the nontrivial limit at the power ε^4:

$$\begin{aligned}
\lim_{\varepsilon\to 0} D\varepsilon^{-4} &= \frac{4}{3}x^4 - \frac{8}{3}ix^3 + 2x^2 - 2ix - \frac{1}{4} - 4(t-\hat{t}_0)^2 - 4\check{x}_0(i-2x),\\
\lim_{\varepsilon\to 0} N_u e^{it}\varepsilon^{-4} &= -\frac{8}{3}x^3 + 4ix^2 + 2x + i - 4i(t-\hat{t}_0)(i-2x) + 8\check{x}_0,\\
\lim_{\varepsilon\to 0} N_v e^{-it}\varepsilon^{-4} &= -\frac{8}{3}x^3 + 4ix^2 + 2x + i + 4i(t-\hat{t}_0)(i-2x) + 8\check{x}_0,
\end{aligned}$$

which yields the explicit expressions (2.20) and (2.21) from the quotients given by (2.17).

B. Computations of (2.22) and (2.23)

We substitute the phase shift (4.1) into $n_1^{(+)}(\lambda_0)$ and $(n_1^{(+)})'(\lambda_0)$ given in (3.29) and (3.31). These expressions define the eigenvector and the generalized eigenvector of the linear systems (2.18) and (2.19) for $\zeta_0 = e^{\frac{i\gamma}{2}}$ by (3.28) and (3.30) respectively. By using $\gamma := \pi - \varepsilon$ and expanding in powers of ε, we derive (2.22) and (2.23).

After the substitution (4.1), we rewrite K_{22} and $-\bar{C}_0\bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)}(K_{22}b_1 + K_{12}b_2)$ in (3.29) as follows:

$$K_{22} = 1 + (\sin\gamma)e^{-2x\sin\gamma}\big(\cot\gamma + 2(x-\hat{x}_0)\cos\gamma - 2i(t-\hat{t}_0)\sin\gamma\big)$$

and

$$-\bar{C}_0\bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)}(K_{22}b_1 + K_{12}b_2) = -e^{-x\sin\gamma - it\cos\gamma - \frac{3i\gamma}{2}} \times\left(e^{i\gamma} + 2\sin\gamma\left[-\cot\gamma + (x-\hat{x}_0)\cos\gamma - i(t-\hat{t}_0)\sin\gamma + \frac{i}{2}\right] - e^{-2x\sin\gamma - 2i\gamma}\right).$$

Expansion as $\varepsilon \to 0$ gives nonzero terms at powers ε^1, and ε^2.

- The coefficients of K_{22}:

$$\begin{aligned} \varepsilon^1 &: \quad 2\hat{x}_0, \\ \varepsilon^2 &: \quad 2x^2 - 2i(t-\hat{t}_0) + \frac{1}{2} - 4x\hat{x}_0. \end{aligned}$$

- The coefficients of $-\bar{C}_0\bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)}(K_{22}b_1 + K_{12}b_2)e^{it\cos\gamma}$:

$$\begin{aligned} \varepsilon^1 &: \quad -2i\hat{x}_0, \\ \varepsilon^2 &: \quad 2ix^2 + 4x - 2(t-\hat{t}_0) - \frac{3}{2}i + 3\hat{x}_0 + 2ix\hat{x}_0. \end{aligned}$$

Rescaling $\hat{x}_0 = \check{x}_0\varepsilon^2$ yields the nontrivial limit at the power ε^2:

$$\lim_{\varepsilon\to 0} K_{22}\varepsilon^{-2} = 2x^2 - 2i(t-\hat{t}_0) + \frac{1}{2}$$

and

$$\lim_{\varepsilon\to 0}\left(-\bar{C}_0\bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)}(K_{22}b_1 + K_{12}b_2)e^{-it}\right)\varepsilon^{-2} = -2ix^2 - 4x + 2(t-\hat{t}_0) + \frac{3}{2}i.$$

By using (3.28) and (3.29), taking the limit

$$\psi_0 := \lim_{\varepsilon\to 0}\varepsilon^2\psi_1^{(+)}(\zeta_0),$$

we obtain (2.22) with

$$\mathfrak{n}_0 := \lim_{\varepsilon\to 0}\varepsilon^2 n_1^{(+)}(\lambda_0)$$

given below (2.22) in the explicit form.

To obtain a nontrivial limit for the generalized eigenvector, we rewrite the expression (3.30) in the equivalent form:

$$(\psi_1^{(+)})'(\zeta_0) - 2(\lambda_0 - \bar{\lambda}_0)^{-1}\psi_1^{(+)}(\zeta_0) = 2\zeta_0 e^{i\theta(\lambda_0)}[T(v,\zeta_0)]^{-1}\left[(n_1^{(+)})'(\lambda_0) + \left(i\theta'(\lambda_0) - 2(\lambda_0-\bar{\lambda}_0)^{-1}\right)n_1^{(+)}(\lambda_0)\right] - \zeta_0^{-1}\partial_\zeta T(v,\zeta)\psi_1^{(+)}(\zeta_0),$$

After the substitution (4.1), we rewrite $-K_{21}$ and $\bar{C}_0\bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)}(K_{11}b_2 + K_{21}b_1)$ in (3.31) as follows:

$$-K_{21} = -2ie^{-2x\sin\gamma - i\gamma}\left[\sin\gamma\left[(x-\hat{x}_0)^2(\cos\gamma)^2 + (t-\hat{t}_0)^2(\sin\gamma)^2\right] + i(t-\hat{t}_0)\sin\gamma\cos\gamma + \frac{\sin\gamma}{4}\right]$$

and

$$\bar{C}_0\bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)}(K_{11}b_2 + K_{21}b_1) = -ie^{-x\sin\gamma - it\cos\gamma - \frac{3i}{2}\gamma} \times\left((x-\hat{x}_0)\cos\gamma - i(t-\hat{t}_0)\sin\gamma + \frac{i}{2} + e^{-2x\sin\gamma - 3i\gamma}\left[(x-\hat{x}_0)\cos\gamma + i(t-\hat{t}_0)\sin\gamma + \frac{i}{2}\right]\right).$$

Expansion as $\varepsilon \to 0$ gives nonzero terms at powers ε^0, ε^1, and ε^2 for both components of the numerator of $(n_1^{(+)})'(\lambda_0) + \left(i\theta'(\lambda_0) - 2(\lambda_0 - \bar{\lambda}_0)^{-1}\right)n_1^{(+)}(\lambda_0)$.

- The coefficients of $-K_{21} + \big(i\theta'(\lambda_0) - 2(\lambda_0 - \bar{\lambda}_0)^{-1}\big)K_{22}$:

$$\begin{aligned}
\varepsilon^0: &\quad -2i\hat{x}_0,\\
\varepsilon^1: &\quad ix\hat{x}_0 + 2i\hat{x}_0^2,\\
\varepsilon^2: &\quad \frac{4}{3}i\hat{x}_0 + x(t-\hat{t}_0) - \frac{1}{3}ix^3 + 2ix^2\hat{x}_0 - \frac{4}{3}ix + \hat{x}_0 t + 3x\hat{x}_0 - 4ix\hat{x}_0^2 - 2x^2 - 2\hat{x}_0^2 - \frac{1}{2} + 2i(t-\hat{t}_0).
\end{aligned}$$

- The coefficients of

$$\bar{C}_0\bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)}\big[K_{11}b_2 + K_{21}b_1 - \big(i\theta'(\lambda_0) - 2(\lambda_0-\bar{\lambda}_0)^{-1}\big)(K_{22}b_1 + K_{12}b_2)\big]e^{it\cos\gamma}:$$

$$\begin{aligned}
e^{it}\varepsilon^2: &\quad -2\hat{x}_0,\\
e^{it}\varepsilon^3: &\quad 6i\hat{x}_0 - 5x\hat{x}_0,\\
e^{it}\varepsilon^4: &\quad ix(t-\hat{t}_0) - it\hat{x}_0 - ix^2 + \frac{5}{4}i - \frac{1}{3}x^3 - \frac{15}{4}x + 3(t-\hat{t}_0) + \frac{155}{12}\hat{x}_0 - 6x^2\hat{x}_0 - it\hat{x}_0 + \frac{35}{2}ix\hat{x}_0.
\end{aligned}$$

Rescaling $\hat{x}_0 = \check{x}_0\varepsilon^2$ yields the nontrivial limit at the power ε^2:

$$\lim_{\varepsilon\to 0}\big[-K_{21} + \big(i\theta'(\lambda_0) - 2(\lambda_0-\bar{\lambda}_0)^{-1}\big)K_{22}\big]\varepsilon^{-2} = -\frac{1}{3}ix^3 - 2x^2 - \frac{4}{3}ix + x(t-\hat{t}_0) + 2i(t-\hat{t}_0) - \frac{1}{2} - 2i\check{x}_0$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\bar{C}_0\bar{\lambda}_0 e^{-2i\theta(\bar{\lambda}_0)}\big[K_{11}b_2 + K_{21}b_1 - \big(i\theta'(\lambda_0) - 2(\lambda_0-\bar{\lambda}_0)^{-1}\big)(K_{22}b_1 + K_{12}b_2)\big]e^{-it}\varepsilon^{-2}\\
&\quad = -\frac{1}{3}x^3 - ix^2 - \frac{15}{4}x + ix(t-\hat{t}_0) + \frac{5}{4}i + 3(t-\hat{t}_0) - 2\check{x}_0.
\end{aligned}$$

By using (3.30) and (3.31), taking the limit

$$\psi_1 := \lim_{\varepsilon\to 0}\varepsilon^2\Big[(\psi_1^{(+)})'(\zeta_0) - 2(\lambda_0-\bar{\lambda}_0)^{-1}\psi_1^{(+)}(\zeta_0)\Big],$$

we obtain (2.23) with

$$\mathfrak{n}_1 := \lim_{\varepsilon\to 0}\varepsilon^2\Big[(n_1^{(+)})'(\lambda_0) + \big(i\theta'(\lambda_0) - 2(\lambda_0-\bar{\lambda}_0)^{-1}\big)n_1^{(+)}(\lambda_0)\Big]$$

given below (2.23) in the explicit form.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Zhi-Qiang Li: Formal analysis (equal); Investigation (equal); Writing – original draft (equal). **Dmitry E. Pelinovsky**: Conceptualization (equal); Investigation (equal); Supervision (equal); Validation (equal); Writing – review & editing (equal). **Shou-Fu Tian**: Project administration (equal); Supervision (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

APPENDIX A: SINGLE-SOLITON FROM A SIMPLE POLE

Here we consider solutions of the normalized RH problem for the reflectionless potential $r_\pm(\lambda) \equiv 0$ for $\lambda \in \mathbb{R}$ with a pair of simple poles of $M(\lambda)$ at $\lambda_0 \in \mathbb{C}^+$ and $\bar{\lambda}_0 \in \mathbb{C}^-$. The normalized RH problem for $M(\lambda) = (P^\infty)^{-1}P(\lambda)$ can be rewritten in the form:

RH problem. Find a complex-valued analytic function $M(\lambda)$ in $\mathbb{C}\backslash\{\mathbb{R}\cup\{\lambda_0,\bar{\lambda}_0\}\}$ with the following properties:

- $M(\lambda)$ has simple poles at $\lambda_0 \in \mathbb{C}^+$ and $\bar{\lambda}_0 \in \mathbb{C}^-$ with the normalization
$$\text{Res}_{\lambda=\lambda_0} M = (P^\infty)^{-1}\left(\vec{\mathbf{0}},\ \frac{n_2^{(-)}(\lambda_0)}{\alpha'(\lambda_0)}\right), \quad \text{Res}_{\lambda=\bar{\lambda}_0} M = (P^\infty)^{-1}\left(\frac{n_1^{(-)}(\bar{\lambda}_0)}{\bar{\alpha}'(\bar{\lambda}_0)}\quad \vec{\mathbf{0}}\right),$$
where $\vec{\mathbf{0}}$ is the 2-by-1 null vector.
- $M(\lambda) \to \mathbb{I}$ as $|\lambda| \to \infty$, where $\mathbb{I}$ is the 2-by-2 identity matrix.
- $M(\lambda)$ is continuous on both sides of $\mathbb{R}$ with $M_\pm(\lambda) := \lim_{\text{Im}(\lambda)\to\pm 0} M(\lambda)$ satisfying
$$M_+(\lambda) = M_-(\lambda), \quad \lambda \in \mathbb{R}.$$

The solution of the RH problem is immediately given by

$$M(x,t,\lambda) = \mathbb{I} + \frac{\text{Res}_{\lambda=\lambda_0} M}{\lambda - \lambda_0} + \frac{\text{Res}_{\lambda=\bar{\lambda}_0} M}{\lambda - \bar{\lambda}_0}. \tag{A1}$$

In order to compute the residue terms, we note from (2.11) that λ_0 is a simple zero of $\alpha(\lambda)$ extended to $\mathbb{C}^+$ by Lemma 2.2. Since it follows from (2.2) that

$$\alpha(\lambda) = a(\zeta) = \det\left(\psi_1^{(+)}(\zeta), \psi_2^{(-)}(\zeta)\right),$$

we define $\zeta_0 := \sqrt{\lambda_0}$ and a constant $b_0 \in \mathbb{C}$ such that the columns of $\psi^{(\pm)}(\zeta)$ satisfying (2.1) are related at $\zeta = \zeta_0$ by

$$\psi_2^{(-)}(\zeta_0) = b_0 \psi_1^{(+)}(\zeta_0). \tag{A2}$$

Since $\lambda_0 \in \mathbb{C}^+$, it follows from (2.1) and (A2) that $\psi_2^{(-)}(\zeta_0)$ decays to zero exponentially fast both as $x \to \pm\infty$. Hence it is the eigenvector of the linear system (1.2) for $\zeta = \zeta_0$.

By using the transformation (2.4), we can rewrite (A2) in the form:

$$n_2^{(-)}(\lambda_0) = b_0 \zeta_0^{-1} n_1^{(+)}(\lambda_0) e^{2i\theta(\lambda_0)}, \tag{A3}$$

where $\theta(\lambda)$ is given by (2.8). By using (A3), we compute the residue term as follows

$$\text{Res}_{\lambda=\lambda_0} P = \left(\vec{\mathbf{0}} \quad \frac{n_2^{(-)}(\lambda_0)}{\alpha'(\lambda_0)}\right) = \left(\vec{\mathbf{0}} \quad \frac{b_0}{\zeta_0 \alpha'(\lambda_0)} n_1^{(+)}(\lambda_0) e^{2i\theta(\lambda_0)}\right). \tag{A4}$$

By using the symmetry condition (2.3), we have

$$\psi_1^{(\pm)}(\zeta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\psi}_2^{(\pm)}(\zeta), \quad \psi_2^{(\pm)}(\zeta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{\psi}_1^{(\pm)}(\zeta),$$

then (A2) can be transformed into

$$\psi_1^{(-)}(\bar{\zeta}_0) = -\bar{b}_0 \psi_2^{(+)}(\bar{\zeta}_0).$$

Using the transformation (2.4), we obtain

$$n_1^{(-)}(\bar{\lambda}_0) = -\bar{b}_0 \bar{\zeta}_0 n_2^{(+)}(\bar{\lambda}_0) e^{-2i\theta(\bar{\lambda}_0)}, \tag{A5}$$

from which we compute the other residue term by using (A5):

$$\text{Res}_{\lambda=\bar{\lambda}_0} P = \left(\frac{n_1^{(-)}(\bar{\lambda}_0)}{\bar{\alpha}'(\bar{\lambda}_0)} \quad \vec{\mathbf{0}}\right) = \left(-\frac{\bar{b}_0\bar{\zeta}_0}{\bar{\alpha}'(\bar{\lambda}_0)} n_2^{(+)}(\bar{\lambda}_0) e^{-2i\theta(\bar{\lambda}_0)} \quad \vec{\mathbf{0}}\right), \tag{A6}$$

We recall that $M(\lambda) = [P^{\infty}]^{-1}P(\lambda)$ with $P^{\infty} = \text{diag}\,(n_1^{+\infty}, n_2^{+\infty})$ with $n_2^{+\infty} = \overline{n_1^{+\infty}}$, see (2.12). Using the first column of (A1) at $\lambda = \lambda_0$ due to (A4) and the second column of (A1) at $\lambda = \bar{\lambda}_0$ due to (A6), we obtain a closed system of linear algebraic equations:

$$n_1^{(+)}(\lambda_0) = n_1^{+\infty} e_1 - \frac{\bar{\lambda}_0 \bar{c}_0}{\lambda_0 - \bar{\lambda}_0} n_2^{(+)}(\bar{\lambda}_0) e^{-2i\theta(\bar{\lambda}_0)} \tag{A7}$$

and

$$n_2^{(+)}(\bar{\lambda}_0) = n_2^{+\infty} e_2 + \frac{c_0}{\bar{\lambda}_0 - \lambda_0} n_1^{(+)}(\lambda_0) e^{2i\theta(\lambda_0)}, \tag{A8}$$

where $e_1 = (1,0)^T$, $e_2 = (0,1)^T$, and

$$c_0 := \frac{b_0}{\zeta_0 \alpha'(\zeta_0)}.$$

Then, from (A7) and (A8), we have

$$n_1^{(+)}(\lambda_0) = n_1^{+\infty} e_1 - \frac{\bar{\lambda}_0 \bar{c}_0}{\lambda_0 - \bar{\lambda}_0} n_2^{+\infty} e^{-2i\theta(\bar{\lambda}_0)} e_2 + \frac{\bar{\lambda}_0 |c_0|^2}{(\lambda_0 - \bar{\lambda}_0)^2} e^{2i\theta(\lambda_0) - 2i\theta(\bar{\lambda}_0)} n_1^{(+)}(\lambda_0),$$
$$n_2^{(+)}(\bar{\lambda}_0) = n_2^{+\infty} e_2 + \frac{c_0}{\bar{\lambda}_0 - \lambda_0} n_1^{+\infty} e^{2i\theta(\lambda_0)} e_1 + \frac{\bar{\lambda}_0 |c_0|^2}{(\lambda_0 - \bar{\lambda}_0)^2} e^{2i\theta(\lambda_0) - 2i\theta(\bar{\lambda}_0)} n_2^{(+)}(\bar{\lambda}_0).$$

By using (2.13), we obtain the explicit solutions to the MTM system (1.1) in the form

$$u = \lim_{\lambda \to 0} \overline{M}_{12}(\lambda) = -\frac{\bar{c}_0}{\bar{\lambda}_0} n_1^{+\infty} \overline{n_{11}^{(+)}(\lambda_0)} e^{-2i\theta(\bar{\lambda}_0)} = -\frac{\bar{c}_0 e^{-2i\theta(\bar{\lambda}_0)}}{\bar{\lambda}_0 - \frac{|\lambda_0|^2 |c_0|^2}{(\lambda_0 - \bar{\lambda}_0)^2} e^{2i\theta(\lambda_0) - 2i\theta(\bar{\lambda}_0)}}$$

and

$$v = \lim_{\lambda \to 0} M_{21}(\lambda) = \bar{c}_0 n_1^{+\infty}(x) n_{22}^{(+)}(\bar{\lambda}_0) e^{-2i\theta(\bar{\lambda}_0)} = \frac{\bar{c}_0 e^{-2i\theta(\bar{\lambda}_0)}}{1 - \frac{\bar{\lambda}_0 |c_0|^2}{(\lambda_0 - \bar{\lambda}_0)^2} e^{2i\theta(\lambda_0) - 2i\theta(\bar{\lambda}_0)}}.$$

To simplify the expressions for the single-soliton solution (u,v), we pick $\lambda_0 = e^{i\gamma}$ with $\gamma \in (0,\pi)$ on $\mathbb{S}^1 \cap \mathbb{C}^+$. A more general solution can be obtained with the Lorentz symmetry (2.14). If $\lambda_0 = e^{i\gamma}$, we obtain from (2.8) that

$$2i\theta(\lambda_0) = -\alpha x + i\beta t,$$

where $\alpha = \sin\gamma$ and $\beta = \cos\gamma$. In addition, we choose

$$c_0 = 2i\,\sin\,\gamma e^{\frac{i\gamma}{2}}$$

and obtain the single-soliton solution in the form

$$u(x,t) = 2i\alpha \frac{e^{-\alpha x - i\beta t + \frac{i\gamma}{2}}}{1 + e^{-2\alpha x + i\gamma}} = i\alpha\,\text{sech}\left(\alpha x - \frac{i\gamma}{2}\right) e^{-i\beta t} \tag{A9}$$

and

$$v(x,t) = -2i\alpha \frac{e^{-\alpha x - i\beta t - \frac{i\gamma}{2}}}{1 + e^{-2\alpha x - i\gamma}} = -i\alpha\,\text{sech}\left(\alpha x + \frac{i\gamma}{2}\right) e^{-i\beta t}, \tag{A10}$$

which coincides with (2.16) since $\alpha = \sin\gamma$ and $\beta = \cos\gamma$. A more general solution with two translational parameters $x_0, t_0 \in \mathbb{R}$ can be obtained by using the symmetries (2.15) or by introducing two translational parameters in the expression for $c_0 \in \mathbb{C}$.

Finally, we write the explicit form of the eigenvector $\psi_1^{(+)}(\zeta_0)$, see (A2), which satisfies (2.18) with (u,v) given by (A9) and (A10) and with $\zeta_0 = e^{\frac{i\gamma}{2}}$. By using the transformation (2.4), we write

$$\psi_1^{(+)}(\zeta_0) = e^{i\theta(\lambda_0)} [T(v,\zeta_0)]^{-1} n_1^{(+)}(\lambda_0),$$

where

$$[T(v,\zeta_0)]^{-1} = \begin{pmatrix} 1 & 0 \\ -e^{-\frac{i\gamma}{2}} v & e^{-\frac{i\gamma}{2}} \end{pmatrix}$$

and

$$\begin{aligned} n_1^{(+)}(\lambda_0) &= \frac{n_1^{+\infty} e_1 - \frac{\bar\lambda_0 \bar c_0}{\lambda_0 - \bar\lambda_0} n_2^{+\infty} e^{-2i\theta(\bar\lambda_0)} e_2}{1 - \frac{\bar\lambda_0 |c_0|^2}{(\lambda_0-\bar\lambda_0)^2} e^{2i\theta(\lambda_0) - 2i\theta(\bar\lambda_0)}} \\ &= \frac{1}{1+e^{-2\alpha x - i\gamma}} \begin{pmatrix} n_1^{+\infty} \\ e^{-\alpha x - i\beta t - \frac{3i\gamma}{2}} n_2^{+\infty} \end{pmatrix} \\ &= \frac{1}{2}\operatorname{sech}\left(\alpha x + \frac{i\gamma}{2}\right) \begin{pmatrix} e^{\alpha x + \frac{i\gamma}{2} + \frac{i}{4}\int_{+\infty}^x (|u|^2+|v|^2)dx} \\ e^{-i\beta t - i\gamma - \frac{i}{4}\int_{+\infty}^x (|u|^2+|v|^2)dx} \end{pmatrix}. \end{aligned}$$

We note that

$$\frac{1}{4}\int_{\mathbb{R}} (|u|^2 + |v|^2)dx = \int_{\mathbb{R}} \frac{\sin^2\gamma}{\cosh(2\sin\gamma\, x) + \cos\gamma} dx = \gamma.$$

Hence we can write

$$e^{i\theta(\lambda_0)} n_1^{(+)}(\lambda_0) = \frac{1}{2}\operatorname{sech}\left(\alpha x + \frac{i\gamma}{2}\right) \begin{pmatrix} e^{\frac{1}{2}\alpha x + \frac{i}{2}\beta t + \frac{i\gamma}{2} + \frac{i}{4}\int_{+\infty}^x (|u|^2+|v|^2)dx} \\ e^{-\frac{1}{2}\alpha x - \frac{i}{2}\beta t - \frac{i}{4}\int_{-\infty}^x (|u|^2+|v|^2)dx} \end{pmatrix}, \tag{A11}$$

which decays exponentially to 0 as $x \to \pm\infty$. Since $[T(v,\zeta_0)]^{-1}$ is bounded, then $\psi_1^{(+)}(\zeta_0) \in H^1(\mathbb{R},\mathbb{C}^2)$ is an exponentially decaying eigenvector of the linear system (2.18).

The algebraic soliton appears in the singular limit $\gamma \to \pi$, where $\zeta_0 \to i$. The simple eigenvalue $\zeta_0 = i$ is embedded into the continuous spectrum of the Lax system (1.2), which is located on $\mathbb{R} \cup (i\mathbb{R})$. By writing $\gamma := \pi - \epsilon$ and taking the limit $\epsilon \to 0$ in (A9) and (A10), we obtain

$$u(x,t) = \frac{2i}{1-2ix} e^{it}, \quad v(x,t) = -\frac{2i}{1+2ix} e^{it}. \tag{A12}$$

The eigenvector ψ_0 for the simple embedded eigenvalue $\zeta_0 = i$ of the linear system (2.18) with (u,v) given by (A12) is obtained from (A11) in the limit $\epsilon \to 0$ in the explicit form:

$$\psi_0 := \lim_{\epsilon\to 0} \epsilon \psi_1^{(+)}(\zeta_0) = \frac{1}{1+2ix} \begin{pmatrix} 1 & 0 \\ iv & -i \end{pmatrix} \begin{pmatrix} e^{-\frac{it}{2} + i\arctan(2x)} \\ e^{\frac{it}{2} - \frac{i\pi}{2} - i\arctan(2x)} \end{pmatrix}, \tag{A13}$$

where we have used the elementary integral

$$\frac{1}{4}\int_{-\infty}^x (|u|^2 + |v|^2)dx = \int_{-\infty}^x \frac{2}{1+4x^2}dx = \frac{\pi}{2} + \arctan(2x).$$

Based on the explicit expression (A13), we confirm that $\psi_0 \in H^1(\mathbb{R},\mathbb{C}^2)$ is an algebraically decaying eigenvector of the linear system (2.18) such that $|\psi_0(x)| = \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$.

APPENDIX B: EXPONENTIAL DOUBLE-SOLITONS IN THE BILINEAR HIROTA METHOD

Here we obtain the exponential double-soliton solutions by using the bilinear Hirota method developed in Ref. 4. To proceed with computations, we use the parameterization from Ref. 8 and write the general exponential two-soliton solutions in the form:

$$u = \frac{g}{f}, \quad v = \frac{h}{f}, \tag{B1}$$

where

$$f = 1 + e^{-2\xi_1 - i\gamma_1} + e^{-2\xi_2 - i\gamma_2} + A_{12} e^{-2\xi_1 - 2\xi_2 - i\gamma_1 - i\gamma_2} - 4\sqrt{\delta_1 \delta_2} \sin\gamma_1 \sin\gamma_2 e^{-\xi_1 - \xi_2 - \frac{i}{2}\gamma_1 - \frac{i}{2}\gamma_2}$$
$$\times \left[\frac{\delta_1 e^{-i(\eta_1 - \eta_2)}}{\left(\delta_1 e^{-\frac{i}{2}(\gamma_1+\gamma_2)} - \delta_2 e^{\frac{i}{2}(\gamma_1+\gamma_2)}\right)^2} + \frac{\delta_2 e^{i(\eta_1 - \eta_2)}}{\left(\delta_1 e^{\frac{i}{2}(\gamma_1+\gamma_2)} - \delta_2 e^{-\frac{i}{2}(\gamma_1+\gamma_2)}\right)^2} \right],$$

$$h = -\bar{\alpha}_1 e^{-\xi_1 - i\eta_1} \left[1 + \left(\frac{p_1 - p_2}{p_1 + \bar{p}_2} \right)^2 e^{-2\xi_2 + i\gamma_2} \right] - \bar{\alpha}_2 e^{-\xi_2 - i\eta_2} \left[1 + \left(\frac{p_1 - p_2}{\bar{p}_1 + p_2} \right)^2 e^{-2\xi_1 + i\gamma_1} \right],$$

and

$$g = \frac{i\bar{\alpha}_1}{p_1} e^{-\xi_1 - i\eta_1} \left[1 + \left(\frac{p_1 - p_2}{p_1 + \bar{p}_2} \right)^2 e^{-2\xi_2 + 3i\gamma_2} \right] + \frac{i\bar{\alpha}_2}{p_2} e^{-\xi_2 - i\eta_2} \left[1 + \left(\frac{p_1 - p_2}{\bar{p}_1 + p_2} \right)^2 e^{-2\xi_1 + 3i\gamma_1} \right],$$

with arbitrary parameters $\gamma_j \in (0, \pi)$, $\delta_j > 0$, $(x_j, t_j) \in \mathbb{R}^2$, and uniquely defined for $j = 1, 2$ as

$$p_j = i\delta_j e^{-i\gamma_j}, \quad \alpha_j = 2\sqrt{\delta_j} \sin\gamma_j e^{\frac{i\gamma_j}{2}},$$

$$\xi_j = \sin\gamma_j \left(\frac{1}{2}(\delta_j + \delta_j^{-1}) x + \frac{1}{2}(\delta_j - \delta_j^{-1}) t + x_j \right),$$
$$\eta_j = \cos\gamma_j \left(\frac{1}{2}(\delta_j - \delta_j^{-1}) x + \frac{1}{2}(\delta_j + \delta_j^{-1}) t + t_j \right),$$

and

$$A_{12} = \left(\frac{\delta_1^2 + \delta_2^2 - 2\delta_1\delta_2 \cos(\gamma_1 - \gamma_2)}{\delta_1^2 + \delta_2^2 - 2\delta_1\delta_2 \cos(\gamma_1 + \gamma_2)} \right)^2.$$

Due to Lorentz transformation (2.14), we can consider the exponential double-solitons with zero speed, for which we take $\delta_1 = \delta_2 = 1$. In addition, we use translational symmetry and replace $e^{-\xi_{1,2}}$ by $e^{-\xi_{1,2}} \sin\left(\frac{\gamma_1+\gamma_2}{2}\right)$ in all expressions.

Considering f, we obtain

$$f = 1 + \sin^2\left(\frac{\gamma_1+\gamma_2}{2}\right)\left[e^{-2\xi_1 - i\gamma_1} + e^{-2\xi_2 - i\gamma_2}\right] + \sin^4\left(\frac{\gamma_1 - \gamma_2}{2}\right) e^{-2\xi_1 - 2\xi_2 - i\gamma_1 - i\gamma_2}$$
$$+ 2 \sin\gamma_1 \sin\gamma_2 e^{-\xi_1 - \xi_2 - \frac{i}{2}\gamma_1 - \frac{i}{2}\gamma_2} \cos(\eta_1 - \eta_2).$$

We now define the small parameter ϵ from $\gamma_1 = \gamma + \epsilon$ and $\gamma_2 = \gamma - \epsilon$ and take the limit $\epsilon \to 0$ for a given $\gamma \in (0, \pi)$. In order to get a nontrivial limit, we also define the translational parameters $x_{1,2}$ from the power series:

$$\xi_1 = (\sin\gamma_1)(x + x_1) = \log(\epsilon) + \alpha(x - x_0) + \epsilon\beta(x - \tilde{x}_0) - \frac{1}{2}\epsilon^2 \alpha(x - \tilde{\tilde{x}}_0) + \mathcal{O}(\epsilon^3),$$
$$\xi_2 = (\sin\gamma_2)(x + x_2) = \log(\epsilon) + \alpha(x - x_0) - \epsilon\beta(x - \tilde{x}_0) - \frac{1}{2}\epsilon^2 \alpha(x - \tilde{\tilde{x}}_0) + \mathcal{O}(\epsilon^3),$$

with new translational parameters $x_0, \tilde{x}_0, \tilde{\tilde{x}}_0 \in \mathbb{R}$ and with $\alpha = \sin\gamma$, $\beta = \cos\gamma$. Similarly, we define the translational parameters $t_{1,2}$ from the power series:

$$\eta_1 = (\cos\gamma_1)(t + t_1) = -\frac{\pi}{2} + \beta(t - t_0) - \epsilon\alpha(t - \tilde{t}_0) - \frac{1}{2}\epsilon^2 \beta(t - \tilde{\tilde{t}}_0) + \mathcal{O}(\epsilon^3),$$
$$\eta_2 = (\cos\gamma_2)(t + t_2) = \frac{\pi}{2} + \beta(t - t_0) + \epsilon\alpha(t - \tilde{t}_0) - \frac{1}{2}\epsilon^2 \beta(t - \tilde{\tilde{t}}_0) + \mathcal{O}(\epsilon^3),$$

with new translational parameters $t_0, \tilde{t}_0, \tilde{\tilde{t}}_0 \in \mathbb{R}$. With these choices, we expand the expression for f in powers of ϵ and obtain the following explicit expression

$$\lim_{\epsilon \to 0} f = 1 + e^{-2\xi - i\gamma}\left[2 + \alpha^2 (2\beta(x - \tilde{x}_0) + i)^2 + 4\alpha^4 (t - \tilde{t}_0)^2\right] + e^{-4\xi - 2i\gamma}, \tag{B2}$$

where $\xi := \alpha(x - x_0)$.

Considering h, we obtain

$$h = -2\sin\gamma_1 \sin\left(\frac{\gamma_1+\gamma_2}{2}\right)e^{-\xi_1 - i\eta_1 - \frac{i\gamma_1}{2}}\left[1 + \left(\frac{e^{-i\gamma_1} - e^{-i\gamma_2}}{e^{-i\gamma_1} - e^{i\gamma_2}}\right)^2 \sin^2\left(\frac{\gamma_1+\gamma_2}{2}\right)e^{-2\xi_2 + i\gamma_2}\right]$$
$$- 2\sin\gamma_2 \sin\left(\frac{\gamma_1+\gamma_2}{2}\right)e^{-\xi_2 - i\eta_2 - \frac{i\gamma_2}{2}}\left[1 + \left(\frac{e^{-i\gamma_1} - e^{-i\gamma_2}}{e^{-i\gamma_2} - e^{i\gamma_1}}\right)^2 \sin^2\left(\frac{\gamma_1+\gamma_2}{2}\right)e^{-2\xi_1 + i\gamma_1}\right],$$

With the choice of the translational parameters above, we expand the expression for h in the powers in ϵ and obtain the following explicit expression

$$\lim_{\epsilon\to 0} h = 4i\alpha^2 e^{-\xi - i\eta - \frac{i\gamma}{2}}\left[-\cot\gamma + \beta(x - \tilde{x}_0) - i\alpha(t - \tilde{t}_0) + \frac{i}{2} - e^{-2\xi - i\gamma}\left(\cot\gamma + \beta(x - \tilde{x}_0) + i\alpha(t - \tilde{t}_0) + \frac{i}{2}\right)\right], \tag{B3}$$

where $\eta := \beta(t - t_0)$.

Considering g, we obtain

$$g = 2\sin\gamma_1 \sin\left(\frac{\gamma_1+\gamma_2}{2}\right)e^{-\xi_1 - i\eta_1 + \frac{i\gamma_1}{2}}\left[1 + \left(\frac{e^{-i\gamma_1} - e^{-i\gamma_2}}{e^{-i\gamma_1} - e^{i\gamma_2}}\right)^2 \sin^2\left(\frac{\gamma_1+\gamma_2}{2}\right)e^{-2\xi_2 + 3i\gamma_2}\right]$$
$$+ 2\sin\gamma_2 \sin\left(\frac{\gamma_1+\gamma_2}{2}\right)e^{-\xi_2 - i\eta_2 + \frac{i\gamma_2}{2}}\left[1 + \left(\frac{e^{-i\gamma_1} - e^{-i\gamma_2}}{e^{-i\gamma_2} - e^{i\gamma_1}}\right)^2 \sin^2\left(\frac{\gamma_1+\gamma_2}{2}\right)e^{-2\xi_1 + 3i\gamma_1}\right].$$

With the same computations, this yields the following explicit expression

$$\lim_{\epsilon\to 0} g = -4i\alpha^2 e^{-\xi - i\eta + \frac{i\gamma}{2}}\left[-\cot\gamma + \beta(x - \tilde{x}_0) - i\alpha(t - \tilde{t}_0) - \frac{i}{2} - e^{-2\xi + i\gamma}\left(\cot\gamma + \beta(x - \tilde{x}_0) + i\alpha(t - \tilde{t}_0) - \frac{i}{2}\right)\right], \tag{B4}$$

The exponential double-solitons are given by the explicit expression (B1) with f, h, and g given by (B2), (B3), and (B4). By using translational symmetry, we can redefine

$$\tilde{x}_0 \to \tilde{x}_0 - \frac{1}{\sin\gamma}, \quad \tilde{t}_0 \to \tilde{t}_0 - \frac{1}{2\sin\gamma}$$

to obtain exactly the same expressions as in Theorem 2.1 for $f = D$, $h = N_v$, and $g = \bar{N}_u$.

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