Stability of Periodic Waves in the Model with Intensity-Dependent Dispersion

[2]Fábio Natali, [1]Dmitry E. Pelinovsky, and [1]Shuoyang Wang

[1]Department of Mathematics & Statistics, McMaster University
[2]Department of Mathematics - State University of Maringá

April 6, 2026

McMaster University recognizes and acknowledges that it is located on the traditional territories of the Mississauga and Haudenosaunee nations, and within the lands protected by the "Dish with One Spoon" wampum agreement.

Introduction

We consider the nonlinear Schrödinger equation with intensity-dependent dispersion (NLS-IDD), written as

$$iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0$$

- $u = u(t, x)$ and $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$.
- Spatially periodic: $u(t, x) = u(t, x + L)$ with period $L > 0$ for any $t \in \mathbb{R}$; denote $\mathbb{T}_L$ as the periodic domain.
- Intensity dependent coefficient: $(1 - |u|^2)$.
- Standing wave solutions $u(t, x) = e^{i\omega t}\phi(x)$ with frequency $\omega \in (0, 1]$ give the spatial system $-(1 - \phi^2)\phi'' + \omega\phi - \phi^3 = 0$.

We study local energetic stability of standing periodic waves with the smooth profiles with respect to periodic perturbations of the same period.

Introduction

We consider the nonlinear Schrödinger equation with intensity-dependent dispersion (NLS-IDD), written as

$$iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0$$

- $u = u(t,x)$ and $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$.
- Spatially periodic: $u(t,x) = u(t,x+L)$ with period $L > 0$ for any $t \in \mathbb{R}$; denote $\mathbb{T}_L$ as the periodic domain.
- Intensity dependent coefficient: $(1 - |u|^2)$.
- Standing wave solutions $u(t,x) = e^{i\omega t}\phi(x)$ with frequency $\omega \in (0,1]$ give the spatial system $-(1-\phi^2)\phi'' + \omega\phi - \phi^3 = 0$.

We study local energetic stability of standing periodic waves with the smooth profiles with respect to periodic perturbations of the same period.

Introduction

We consider the nonlinear Schrödinger equation with intensity-dependent dispersion (NLS-IDD), written as

$$iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0$$

- $u = u(t,x)$ and $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$.
- Spatially periodic: $u(t,x) = u(t,x+L)$ with period $L > 0$ for any $t \in \mathbb{R}$; denote $\mathbb{T}_L$ as the periodic domain.
- Intensity dependent coefficient: $(1 - |u|^2)$.
- Standing wave solutions $u(t,x) = e^{i\omega t}\phi(x)$ with frequency $\omega \in (0,1]$ give the spatial system $-(1-\phi^2)\phi'' + \omega\phi - \phi^3 = 0$.

We study local energetic stability of standing periodic waves with the smooth profiles with respect to periodic perturbations of the same period.

Introduction

We consider the nonlinear Schrödinger equation with intensity-dependent dispersion (NLS-IDD), written as

$$iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0$$

- $u = u(t, x)$ and $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$.
- Spatially periodic: $u(t, x) = u(t, x + L)$ with period $L > 0$ for any $t \in \mathbb{R}$; denote $\mathbb{T}_L$ as the periodic domain.
- Intensity dependent coefficient: $(1 - |u|^2)$.
- Standing wave solutions $u(t, x) = e^{i\omega t}\phi(x)$ with frequency $\omega \in (0, 1]$ give the spatial system $-(1 - \phi^2)\phi'' + \omega\phi - \phi^3 = 0$.

We study local energetic stability of standing periodic waves with the smooth profiles with respect to periodic perturbations of the same period.

Introduction

We consider the nonlinear Schrödinger equation with intensity-dependent dispersion (NLS-IDD), written as

$$\boxed{iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0}$$

- $u = u(t, x)$ and $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$.
- Spatially periodic: $u(t, x) = u(t, x + L)$ with period $L > 0$ for any $t \in \mathbb{R}$; denote $\mathbb{T}_L$ as the periodic domain.
- Intensity dependent coefficient: $(1 - |u|^2)$.
- Standing wave solutions $u(t, x) = e^{i\omega t}\phi(x)$ with frequency $\omega \in (0, 1]$ give the spatial system $-(1 - \phi^2)\phi'' + \omega\phi - \phi^3 = 0$.

We study local energetic stability of standing periodic waves with the smooth profiles with respect to periodic perturbations of the same period.

Introduction

We consider the nonlinear Schrödinger equation with intensity-dependent dispersion (NLS-IDD), written as

$$\boxed{iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0}$$

- $u = u(t, x)$ and $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$.
- Spatially periodic: $u(t, x) = u(t, x + L)$ with period $L > 0$ for any $t \in \mathbb{R}$; denote $\mathbb{T}_L$ as the periodic domain.
- Intensity dependent coefficient: $(1 - |u|^2)$.
- Standing wave solutions $u(t, x) = e^{i\omega t}\phi(x)$ with frequency $\omega \in (0, 1]$ give the spatial system $-(1 - \phi^2)\phi'' + \omega\phi - \phi^3 = 0$.

We study local energetic stability of standing periodic waves with the smooth profiles with respect to periodic perturbations of the same period.

Existence of Periodic Waves

For $\omega \in (0,1)$, the solutions of $-(1-\phi^2)\phi'' + \omega\phi - \phi^3 = 0$ are smooth waves.

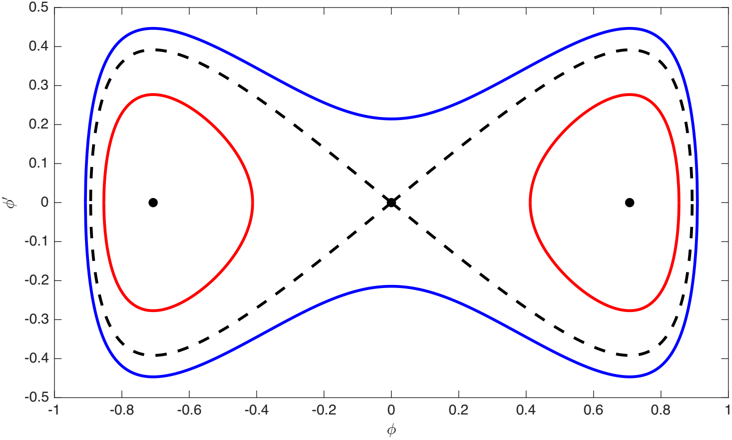

Figure: Phase portrait at $\omega = 0.5$; red orbit: even waves; blue orbit: odd waves; dashed: homoclinic orbit

Existence of Periodic Waves

Theorem (F. Natali-D. Pelinovsky-S. Wang; Nonlinearity, 2026, Preprint)

Consider the system $-(1-\phi^2)\phi'' + \omega\phi - \phi^3 = 0$. *Fix the spatial period* $L > 0$. *for the periodic domain* $\mathbb{T}_L$.

- *For any* $\omega \in (\frac{2\pi^2}{L^2+2\pi^2}, 1)$, *there exists a periodic orbit of system with the* **even** *smooth profile* ϕ *satisfying*

$$\begin{cases} 0 < \phi(x) < 1, & \forall x \in \mathbb{T}_L, \\ \phi(x) = \phi(-x), & \forall x \in \mathbb{T}_L. \end{cases}$$

- *For any* $\omega \in (-\frac{4\pi^2}{L^2}, 1)$, *there exists a periodic orbit of system with the* **odd** *smooth profile* ϕ *satisfying*

$$\begin{cases} -1 < \phi(x) < 1, & \forall x \in \mathbb{T}_L, \\ \phi(x) = -\phi(-x) = \phi\left(\frac{L}{2} - x\right), & \forall x \in \mathbb{T}_L. \end{cases}$$

Work Flow

Our goal is to show energetic stability.

- (Done) Existence of Periodic Waves
- (Next task) Construct period function $T(\mathcal{E}, \omega)$ and analyze to establish monotonicity.
- Use monotonicity to characterize the number of negative eigenvalues of Hessian operator.
- Impose constraint(s) to remove negative directions and leads us to energetic stability.

Monotonicity of Period Function

Write $-(1-\phi^2)\phi'' + \omega\phi - \phi^3 = 0$ using potential function $V(\phi)$ as

$$\phi'' = \frac{(\omega - \phi^2)}{1-\phi^2}\phi = -\frac{dV}{d\phi}, \quad V(\phi) = \frac{1}{2}(\omega - \phi^2) + \frac{1}{2}(1-\omega)\log\frac{1-\omega}{1-\phi^2},$$

and the total energy $E(\phi,\phi') = \frac{1}{2}(\phi')^2 + V(\phi)$ is conserved along each solution.

The two families of periodic orbits correspond to the energy level $\mathcal{E} = E(\phi,\phi')$, and the level of homoclinic orbit is $\mathcal{E}_\omega = V(0)$.

- even waves: $\mathcal{E} \in (0, \mathcal{E}_\omega)$
- odd waves: $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$

The period function of a full closed periodic orbit is defined as

$$T(\mathcal{E},\omega) = \oint \frac{d\phi}{\sqrt{2(\mathcal{E} - V(\phi))}}.$$

Monotonicity of Period Function

Write $-(1-\phi^2)\phi'' + \omega\phi - \phi^3 = 0$ using potential function $V(\phi)$ as

$$\phi'' = \frac{(\omega - \phi^2)}{1-\phi^2}\phi = -\frac{dV}{d\phi}, \quad V(\phi) = \frac{1}{2}(\omega - \phi^2) + \frac{1}{2}(1-\omega)\log\frac{1-\omega}{1-\phi^2},$$

and the total energy $E(\phi, \phi') = \frac{1}{2}(\phi')^2 + V(\phi)$ is conserved along each solution.

The two families of periodic orbits correspond to the energy level $\mathcal{E} = E(\phi, \phi')$, and the level of homoclinic orbit is $\mathcal{E}_\omega = V(0)$.

- even waves: $\mathcal{E} \in (0, \mathcal{E}_\omega)$
- odd waves: $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$

The period function of a full closed periodic orbit is defined as

$$T(\mathcal{E}, \omega) = \oint \frac{d\phi}{\sqrt{2(\mathcal{E} - V(\phi))}}.$$

Monotonicity of Period Function

Write $-(1-\phi^2)\phi'' + \omega\phi - \phi^3 = 0$ using potential function $V(\phi)$ as

$$\phi'' = \frac{(\omega-\phi^2)}{1-\phi^2}\phi = -\frac{dV}{d\phi}, \quad V(\phi) = \frac{1}{2}(\omega-\phi^2) + \frac{1}{2}(1-\omega)\log\frac{1-\omega}{1-\phi^2},$$

and the total energy $E(\phi,\phi') = \frac{1}{2}(\phi')^2 + V(\phi)$ is conserved along each solution.

The two families of periodic orbits correspond to the energy level $\mathcal{E} = E(\phi,\phi')$, and the level of homoclinic orbit is $\mathcal{E}_\omega = V(0)$.

- even waves: $\mathcal{E} \in (0, \mathcal{E}_\omega)$
- odd waves: $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$

The period function of a full closed periodic orbit is defined as

$$T(\mathcal{E},\omega) = \oint \frac{d\phi}{\sqrt{2(\mathcal{E}-V(\phi))}}.$$

Monotonicity of Period Function

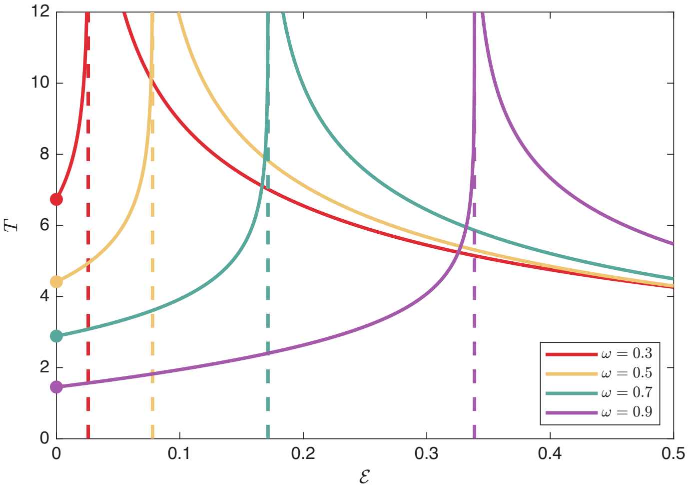

Figure: Period function $T = T(\mathcal{E}, \omega)$; dashed lines: homoclinic orbit.

Monotonicity of Period Function

Theorem (F. Natali-D. Pelinovsky-S. Wang; Nonlinearity, 2026, Preprint)

The period function

$$T(\mathcal{E},\omega) = \oint \frac{d\phi}{\sqrt{2(\mathcal{E} - V(\phi))}}$$

is C^1 for $\mathcal{E} \in (0,\infty)\backslash\mathcal{E}_\omega$ if $\omega \in (0,1)$. For any $\omega \in (0,1)$, the mapping

- $(0,\mathcal{E}_\omega) \ni \mathcal{E} \to T(\mathcal{E},\omega)$ *is monotonically increasing (even waves).*
- $(\mathcal{E}_\omega,\infty) \ni \mathcal{E} \to T(\mathcal{E},\omega)$ *is monotonically decreasing (odd waves).*

Remark

- The proof is explicit analysis of $T(\mathcal{E},\omega)$.
  - Even wave: estimate logarithmic and polynomial functions and then use Chicone's theorem (1987).
  - Odd wave: straightforward estimation by convexity.
- For any spatial period $L > 0$, its energy level is denoted as $\mathcal{E} = \mathcal{E}_L(\omega)$, and the parameter pair $(\mathcal{E}_L(\omega),\omega)$ is computed as the root of $T(\mathcal{E}_L(\omega),\omega) = L$.

Monotonicity of Period Function

Theorem (F. Natali-D. Pelinovsky-S. Wang; Nonlinearity, 2026, Preprint)

The period function

$$T(\mathcal{E},\omega) = \oint \frac{d\phi}{\sqrt{2(\mathcal{E} - V(\phi))}}$$

is C^1 for $\mathcal{E} \in (0,\infty)\backslash\mathcal{E}_\omega$ if $\omega \in (0,1)$. For any $\omega \in (0,1)$, the mapping

- $(0,\mathcal{E}_\omega) \ni \mathcal{E} \to T(\mathcal{E},\omega)$ *is monotonically increasing (even waves).*
- $(\mathcal{E}_\omega,\infty) \ni \mathcal{E} \to T(\mathcal{E},\omega)$ *is monotonically decreasing (odd waves).*

Remark

- The proof is explicit analysis of $T(\mathcal{E},\omega)$.
  - Even wave: estimate logarithmic and polynomial functions and then use Chicone's theorem (1987).
  - Odd wave: straightforward estimation by convexity.
- For any spatial period $L > 0$, its energy level is denoted as $\mathcal{E} = \mathcal{E}_L(\omega)$, and the parameter pair $(\mathcal{E}_L(\omega),\omega)$ is computed as the root of $T(\mathcal{E}_L(\omega),\omega) = L$.

Monotonicity of Period Function

Theorem (F. Natali-D. Pelinovsky-S. Wang; Nonlinearity, 2026, Preprint)

The period function

$$T(\mathcal{E},\omega) = \oint \frac{d\phi}{\sqrt{2(\mathcal{E} - V(\phi))}}$$

is C^1 for $\mathcal{E} \in (0,\infty)\backslash\mathcal{E}_\omega$ if $\omega \in (0,1)$. For any $\omega \in (0,1)$, the mapping

- $(0,\mathcal{E}_\omega) \ni \mathcal{E} \to T(\mathcal{E},\omega)$ *is monotonically increasing (even waves).*
- $(\mathcal{E}_\omega,\infty) \ni \mathcal{E} \to T(\mathcal{E},\omega)$ *is monotonically decreasing (odd waves).*

Remark

- The proof is explicit analysis of $T(\mathcal{E},\omega)$.
  - Even wave: estimate logarithmic and polynomial functions and then use Chicone's theorem (1987).
  - Odd wave: straightforward estimation by convexity.
- For any spatial period $L > 0$, its energy level is denoted as $\mathcal{E} = \mathcal{E}_L(\omega)$, and the parameter pair $(\mathcal{E}_L(\omega),\omega)$ is computed as the root of $T(\mathcal{E}_L(\omega),\omega) = L$.

Approximation of Solutions

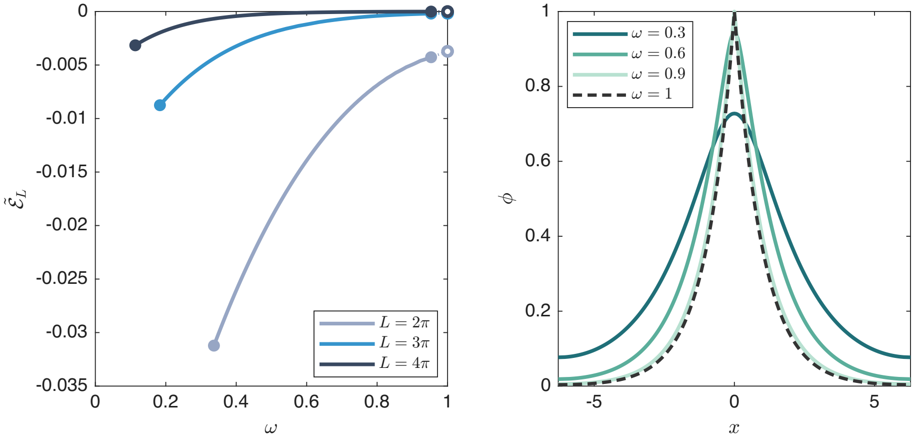

Figure: [Left] the parameter pairs $(\tilde{\mathcal{E}}_L, \omega)$ where $\tilde{\mathcal{E}}_L(\omega) = \mathcal{E}_L(\omega) - \mathcal{E}_\omega$ as the root of $T(\mathcal{E}_L(\omega), \omega) = L$ and the values at $\omega = 1$ are $\tilde{\mathcal{E}}_L = -[2\cosh^2(\frac{L}{2})]^{-1}$; [right] corresponding even periodic wave for $L = 4\pi$.

Approximation of Solutions

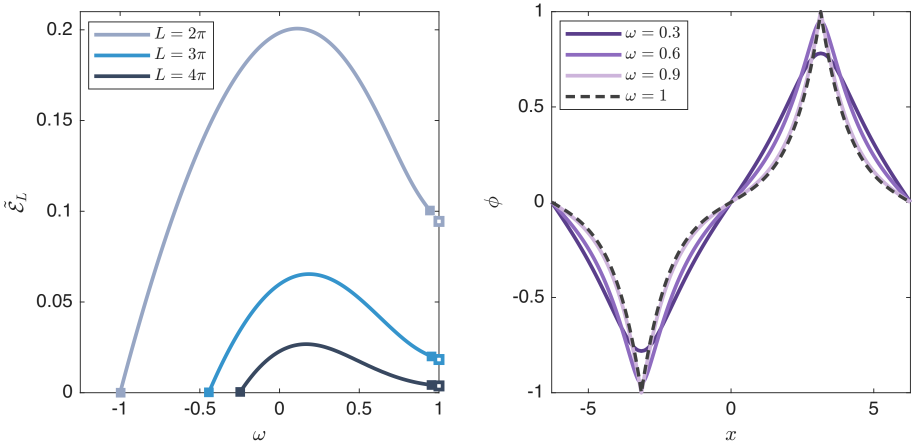

Figure: [Left] the parameter pairs $(\tilde{\mathcal{E}}_L, \omega)$ where $\tilde{\mathcal{E}}_L(\omega) = \mathcal{E}_L(\omega) - \mathcal{E}_\omega$ as the root of $T(\mathcal{E}_L(\omega), \omega) = L$ and the values at $\omega = 1$ are $\tilde{\mathcal{E}}_L = \frac{1}{2}[\sinh^2(\frac{L}{4})]^{-1}$; [right] corresponding odd periodic wave for $L = 4\pi$.

Work Flow

Our goal is to show energetic stability.

- (Done) Existence of Periodic Waves
- (Done) Construct period function $T(\mathcal{E}, \omega)$ and analyze to establish monotonicity.
- (Next task) Use monotonicity to characterize the number of negative eigenvalues of Hessian operator.
- Impose constraint(s) to remove negative directions and leads us to energetic stability.

Energetic Stability

The time-dependent NLS-IDD equation

$$iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{T}_L.$$

admits two conserved quantities in space $\mathcal{X} = \left\{ u \in H^1_{\rm per} : \|u\|_{L^\infty} < 1 \right\}$:

- Energy: $H(u) = \int_{\mathbb{T}_L} \left(|u_x|^2 + |u|^2 + \log\left(1 - |u|^2\right) \right) dx$
- Mass: $Q(u) = -\int_{\mathbb{T}_L} \log\left(1 - |u|^2\right) dx$

Consider the augmented functional $G(u) := H(u) + \omega Q(u)$, so its Hessian operator $\mathcal{L} = G''(u) = H''(u) + \omega Q''(u)$ at the critical point with the profile ϕ is defined as

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}, \qquad \begin{array}{l} \mathcal{L}_+ = -\partial_x^2 + 1 + (\omega - 1)\frac{1+\phi^2}{(1-\phi^2)^2}, \\ \mathcal{L}_- = -\partial_x^2 + 1 + (\omega - 1)\frac{1}{(1-\phi^2)}, \end{array}$$

and we study the spectrum of $\mathcal{L}$ in $L^2_{\rm per} \times L^2_{\rm per}$.

Energetic Stability

The time-dependent NLS-IDD equation

$$iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{T}_L.$$

admits two conserved quantities in space $\mathcal{X} = \left\{u \in H^1_{\rm per} : \|u\|_{L^\infty} < 1\right\}$:

- Energy: $H(u) = \int_{\mathbb{T}_L} \left(|u_x|^2 + |u|^2 + \log\left(1 - |u|^2\right)\right) dx$
- Mass: $Q(u) = -\int_{\mathbb{T}_L} \log\left(1 - |u|^2\right) dx$

Consider the augmented functional $G(u) := H(u) + \omega Q(u)$, so its Hessian operator $\mathcal{L} = G''(u) = H''(u) + \omega Q''(u)$ at the critical point with the profile ϕ is defined as

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}, \qquad \begin{array}{l} \mathcal{L}_+ = -\partial_x^2 + 1 + (\omega - 1)\frac{1+\phi^2}{(1-\phi^2)^2}, \\ \mathcal{L}_- = -\partial_x^2 + 1 + (\omega - 1)\frac{1}{(1-\phi^2)}, \end{array}$$

and we study the spectrum of $\mathcal{L}$ in $L^2_{\rm per} \times L^2_{\rm per}$.

Energetic Stability

The time-dependent NLS-IDD equation

$$iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{T}_L.$$

admits two conserved quantities in space $\mathcal{X} = \left\{ u \in H^1_{\rm per} : \|u\|_{L^\infty} < 1 \right\}$:

- Energy: $H(u) = \int_{\mathbb{T}_L} \left(|u_x|^2 + |u|^2 + \log\left(1 - |u|^2\right) \right) dx$
- Mass: $Q(u) = -\int_{\mathbb{T}_L} \log\left(1 - |u|^2\right) dx$

Consider the augmented functional $G(u) := H(u) + \omega Q(u)$, so its Hessian operator $\mathcal{L} = G''(u) = H''(u) + \omega Q''(u)$ at the critical point with the profile ϕ is defined as

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}, \qquad \begin{aligned} \mathcal{L}_+ &= -\partial_x^2 + 1 + (\omega - 1)\tfrac{1+\phi^2}{(1-\phi^2)^2}, \\ \mathcal{L}_- &= -\partial_x^2 + 1 + (\omega - 1)\tfrac{1}{(1-\phi^2)}, \end{aligned}$$

and we study the spectrum of $\mathcal{L}$ in $L^2_{\rm per} \times L^2_{\rm per}$.

Energetic Stability

For the Hessian operator

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}, \qquad \begin{array}{l} \mathcal{L}_+ = -\partial_x^2 + 1 + (\omega - 1)\frac{1+\phi^2}{(1-\phi^2)^2}, \\ \mathcal{L}_- = -\partial_x^2 + 1 + (\omega - 1)\frac{1}{(1-\phi^2)}, \end{array}$$

$\sigma(\mathcal{L}_\pm)$ consists of real eigenvalues $\lambda_0, \lambda_1, \dots$ and we characterize them by the following definitions.

Definition

- Morse index $n(\mathcal{L}_\pm)$: number of negative eigenvalues.
- Nullity index $z(\mathcal{L}_\pm)$: multiplicity of zero eigenvalue.

We use previous theorem on monotonicity of period function $T(\mathcal{E}, \omega)$ to prove

Proposition

- Even wave: $n(\mathcal{L}_+) = z(\mathcal{L}_+) = z(\mathcal{L}_-) = 1, n(\mathcal{L}_-) = 0$, 1 negative direction
- Odd wave: $n(\mathcal{L}_+) = 2, z(\mathcal{L}_+) = n(\mathcal{L}_-) = z(\mathcal{L}_-) = 1$, 3 negative directions

Energetic Stability

For the Hessian operator

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}, \qquad \begin{aligned} \mathcal{L}_+ &= -\partial_x^2 + 1 + (\omega - 1)\frac{1+\phi^2}{(1-\phi^2)^2}, \\ \mathcal{L}_- &= -\partial_x^2 + 1 + (\omega - 1)\frac{1}{(1-\phi^2)}, \end{aligned}$$

$\sigma(\mathcal{L}_\pm)$ consists of real eigenvalues $\lambda_0, \lambda_1, \dots$ and we characterize them by the following definitions.

Definition

- Morse index $n(\mathcal{L}_\pm)$: number of negative eigenvalues.
- Nullity index $z(\mathcal{L}_\pm)$: multiplicity of zero eigenvalue.

We use previous theorem on monotonicity of period function $T(\mathcal{E}, \omega)$ to prove

Proposition

- Even wave: $n(\mathcal{L}_+) = z(\mathcal{L}_+) = z(\mathcal{L}_-) = 1, n(\mathcal{L}_-) = 0$, 1 negative direction
- Odd wave: $n(\mathcal{L}_+) = 2, z(\mathcal{L}_+) = n(\mathcal{L}_-) = z(\mathcal{L}_-) = 1$, 3 negative directions

Energetic Stability

For the Hessian operator

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}, \qquad \begin{array}{l} \mathcal{L}_+ = -\partial_x^2 + 1 + (\omega - 1)\frac{1+\phi^2}{(1-\phi^2)^2}, \\ \mathcal{L}_- = -\partial_x^2 + 1 + (\omega - 1)\frac{1}{(1-\phi^2)}, \end{array}$$

$\sigma(\mathcal{L}_\pm)$ consists of real eigenvalues $\lambda_0, \lambda_1, \dots$ and we characterize them by the following definitions.

Definition

- Morse index $n(\mathcal{L}_\pm)$: number of negative eigenvalues.
- Nullity index $z(\mathcal{L}_\pm)$: multiplicity of zero eigenvalue.

We use previous theorem on monotonicity of period function $T(\mathcal{E}, \omega)$ to prove

Proposition

- Even wave: $n(\mathcal{L}_+) = z(\mathcal{L}_+) = z(\mathcal{L}_-) = 1, n(\mathcal{L}_-) = 0$, 1 negative direction
- Odd wave: $n(\mathcal{L}_+) = 2, z(\mathcal{L}_+) = n(\mathcal{L}_-) = z(\mathcal{L}_-) = 1$, 3 negative directions

Work Flow

Our goal is to show energetic stability.

- (Done) Existence of Periodic Waves
- (Done) Construct period function $T(\mathcal{E}, \omega)$ and analyze to establish monotonicity.
- (Done) Use monotonicity to characterize the number of negative eigenvalues of Hessian operator.
- (Next Task) Impose constraint(s) to remove negative directions and leads us to energetic stability.

Energetic Stability

The time-dependent NLS-IDD equation

$$iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{T}_L.$$

Write perturbation as $u(t,x) = e^{i\omega t}\left(\phi(x) + p(t,x) + iq(t,x)\right)$, and we consider variation of energy $H(u)$ under fixed mass $Q(u)$, and define real part constraint

$$\langle \phi_0, p \rangle_{L^2_{\rm per}} = 0, \quad \phi_0 \equiv \frac{\phi}{1 - \phi^2}.$$

This constraint removes one negative eigenvalue for $\mathcal{L}_+$.

Proposition

- Even wave: $n(\mathcal{L}_+) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 0$, no negative direction.

Remark The even case gives constraint minimizer. The odd case does not since $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 1, n(\mathcal{L}_-) = 1$, 2 negative directions.

Energetic Stability

The time-dependent NLS-IDD equation

$$iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{T}_L.$$

Write perturbation as $u(t,x) = e^{i\omega t}\left(\phi(x) + p(t,x) + iq(t,x)\right)$, and we consider variation of energy $H(u)$ under fixed mass $Q(u)$, and define real part constraint

$$\langle \phi_0, p \rangle_{L^2_{\rm per}} = 0, \quad \phi_0 \equiv \frac{\phi}{1 - \phi^2}.$$

This constraint removes one negative eigenvalue for $\mathcal{L}_+$.

Proposition

- Even wave: $n(\mathcal{L}_+) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 0$, no negative direction.

Remark The even case gives constraint minimizer. The odd case does not since $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 1, n(\mathcal{L}_-) = 1$, 2 negative directions.

Energetic Stability

The time-dependent NLS-IDD equation

$$iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{T}_L.$$

Write perturbation as $u(t,x) = e^{i\omega t}(\phi(x) + p(t,x) + iq(t,x))$, and we consider variation of energy $H(u)$ under fixed mass $Q(u)$, and define real part constraint

$$\langle \phi_0, p \rangle_{L^2_{\rm per}} = 0, \quad \phi_0 \equiv \frac{\phi}{1 - \phi^2}.$$

This constraint removes one negative eigenvalue for $\mathcal{L}_+$.

Proposition

- Even wave: $n(\mathcal{L}_+) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 0$, no negative direction.

Remark The even case gives constraint minimizer. The odd case does not since $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 1, n(\mathcal{L}_-) = 1$, 2 negative directions.

Energetic Stability

In addition to the constraint $\langle \phi_0, p \rangle_{L^2_{\rm per}} = 0$, we further restrict odd waves to subspace $\mathcal{Y} \subset H^1_{\rm per}$ that contains odd functions with respect to the half-period: $\mathcal{Y} = \left\{ u \in H^1_{\rm per} : u\left(\frac{L}{2} - x\right) = -u\left(x - \frac{L}{2}\right), \forall x \in \mathbb{T}_L \right\}$.

Proposition

- Odd wave: $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = n(\mathcal{L}_-|_{\mathcal{Y}}) = 0$, no negative direction.

Remark

- Even wave: $n(\mathcal{L}_+) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 0$ and $n(\mathcal{L}_-) = 0$; remove **one** negative direction and give local constraint minimizer of $H(u)$ in space $H^1_{\rm per}$.
- Odd wave: $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\mathcal{Y}}) = 1$, $n(\mathcal{L}_-|_{\mathcal{Y}}) = 0 \Rightarrow$ $n(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = 0$, remove **three** negative directions and give local constraint minimizer in subspace $\mathcal{Y} \subset H^1_{\rm per}$.

Energetic Stability

In addition to the constraint $\langle \phi_0, p \rangle_{L^2_{\rm per}} = 0$, we further restrict odd waves to subspace $\mathcal{Y} \subset H^1_{\rm per}$ that contains odd functions with respect to the half-period: $\mathcal{Y} = \left\{ u \in H^1_{\rm per} : u\left(\frac{L}{2} - x\right) = -u\left(x - \frac{L}{2}\right), \forall x \in \mathbb{T}_L \right\}$.

Proposition

- Odd wave: $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = n(\mathcal{L}_-|_{\mathcal{Y}}) = 0$, no negative direction.

Remark

- Even wave: $n(\mathcal{L}_+) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 0$ and $n(\mathcal{L}_-) = 0$; remove **one** negative direction and give local constraint minimizer of $H(u)$ in space $H^1_{\rm per}$.
- Odd wave: $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\mathcal{Y}}) = 1$, $n(\mathcal{L}_-|_{\mathcal{Y}}) = 0 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = 0$, remove **three** negative directions and give local constraint minimizer in subspace $\mathcal{Y} \subset H^1_{\rm per}$.

Energetic Stability

In addition to the constraint $\langle \phi_0, p \rangle_{L^2_{\rm per}} = 0$, we further restrict odd waves to subspace $\mathcal{Y} \subset H^1_{\rm per}$ that contains odd functions with respect to the half-period: $\mathcal{Y} = \left\{ u \in H^1_{\rm per} : u\left(\frac{L}{2} - x\right) = -u\left(x - \frac{L}{2}\right), \forall x \in \mathbb{T}_L \right\}$.

Proposition

- Odd wave: $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = n(\mathcal{L}_-|_{\mathcal{Y}}) = 0$, no negative direction.

Remark

- Even wave: $n(\mathcal{L}_+) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 0$ and $n(\mathcal{L}_-) = 0$; remove **one** negative direction and give local constraint minimizer of $H(u)$ in space $H^1_{\rm per}$.
- Odd wave: $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\mathcal{Y}}) = 1$, $n(\mathcal{L}_-|_{\mathcal{Y}}) = 0 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = 0$, remove **three** negative directions and give local constraint minimizer in subspace $\mathcal{Y} \subset H^1_{\rm per}$.

Energetic Stability

In addition to the constraint $\langle \phi_0, p \rangle_{L^2_{\text{per}}} = 0$, we further restrict odd waves to subspace $\mathcal{Y} \subset H^1_{\text{per}}$ that contains odd functions with respect to the half-period: $\mathcal{Y} = \left\{ u \in H^1_{\text{per}} : u\left(\frac{L}{2} - x\right) = -u\left(x - \frac{L}{2}\right), \forall x \in \mathbb{T}_L \right\}$.

Proposition

- Odd wave: $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = n(\mathcal{L}_-|_{\mathcal{Y}}) = 0$, no negative direction.

Remark

- Even wave: $n(\mathcal{L}_+) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 0$ and $n(\mathcal{L}_-) = 0$; remove **one** negative direction and give local constraint minimizer of $H(u)$ in space H^1_{per}.
- Odd wave: $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\mathcal{Y}}) = 1$, $n(\mathcal{L}_-|_{\mathcal{Y}}) = 0 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = 0$, remove **three** negative directions and give local constraint minimizer in subspace $\mathcal{Y} \subset H^1_{\text{per}}$.

Energetic Stability

In addition to the constraint $\langle \phi_0, p \rangle_{L^2_{\rm per}} = 0$, we further restrict odd waves to subspace $\mathcal{Y} \subset H^1_{\rm per}$ that contains odd functions with respect to the half-period: $\mathcal{Y} = \left\{ u \in H^1_{\rm per} : u\left(\frac{L}{2} - x\right) = -u\left(x - \frac{L}{2}\right), \forall x \in \mathbb{T}_L \right\}$.

Proposition

- Odd wave: $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = n(\mathcal{L}_-|_\mathcal{Y}) = 0$, no negative direction.

Remark

- Even wave: $n(\mathcal{L}_+) = 1 \Rightarrow n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 0$ and $n(\mathcal{L}_-) = 0$; remove **one** negative direction and give local constraint minimizer of $H(u)$ in space $H^1_{\rm per}$.
- Odd wave: $n(\mathcal{L}_+) = 2, n(\mathcal{L}_-) = 1 \Rightarrow n(\mathcal{L}_+|_\mathcal{Y}) = 1$, $n(\mathcal{L}_-|_\mathcal{Y}) = 0 \Rightarrow$ $n(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = 0$, remove **three** negative directions and give local constraint minimizer in subspace $\mathcal{Y} \subset H^1_{\rm per}$.

Energetic Stability

Theorem (F. Natali-D. Pelinovsky-S. Wang; Nonlinearity, 2026, Preprint)

Fix $L > 0$ and consider the system $-(1-\phi^2)\phi'' + \omega\phi - \phi^3 = 0$. The profile $\phi \in H^1_{\rm per}$ is a C^1 function of ω.

- **Even wave**: *For any $\omega \in \left(\frac{2\pi^2}{L^2+2\pi^2}, 1\right)$, the even wave satisfying the system is a local minimizer of energy $H(u)$ for a fixed mass $Q(u)$ in $H^1_{\rm per}$, which is only degenerate by the translational and rotational symmetries, if and only if the mapping $\omega \to Q(\phi)$ is monotonically increasing.*
- **Odd wave**: *For any $\omega \in (-\frac{4\pi^2}{L^2}, 1)$, the odd wave with the profile ϕ satisfying the system is a local minimizer of energy $H(u)$ for a fixed mass $Q(u)$ in $\mathcal{Y} \subset H^1_{\rm per}$, where $\mathcal{Y} = \left\{u \in H^1_{\rm per} : u\left(\frac{L}{2} - x\right) = -u\left(x - \frac{L}{2}\right), \forall x \in \mathbb{T}\right\}$, which is only degenerate by the rotational symmetry, if and only if the mapping $\omega \to Q(\phi)$ is monotonically increasing.*

Remark The proof has two steps. (1) The Morse and nullity indices of Hessian are characterized by a criterion developed by Neves (2009). (2) Constraints are used to connect to the sign of $Q'(\omega)$.

Energetic Stability

Theorem (F. Natali-D. Pelinovsky-S. Wang; Nonlinearity, 2026, Preprint)

Fix $L > 0$ and consider the system $-(1-\phi^2)\phi'' + \omega\phi - \phi^3 = 0$. The profile $\phi \in H^1_{\rm per}$ is a C^1 function of ω.

- **Even wave**: *For any $\omega \in \left(\frac{2\pi^2}{L^2+2\pi^2}, 1\right)$, the even wave satisfying the system is a local minimizer of energy $H(u)$ for a fixed mass $Q(u)$ in $H^1_{\rm per}$, which is only degenerate by the translational and rotational symmetries, if and only if the mapping $\omega \to Q(\phi)$ is monotonically increasing.*
- **Odd wave**: *For any $\omega \in (-\frac{4\pi^2}{L^2}, 1)$, the odd wave with the profile ϕ satisfying the system is a local minimizer of energy $H(u)$ for a fixed mass $Q(u)$ in $\mathcal{Y} \subset H^1_{\rm per}$, where $\mathcal{Y} = \left\{u \in H^1_{\rm per} : u\left(\frac{L}{2} - x\right) = -u\left(x - \frac{L}{2}\right), \forall x \in \mathbb{T}\right\}$, which is only degenerate by the rotational symmetry, if and only if the mapping $\omega \to Q(\phi)$ is monotonically increasing.*

Remark The proof has two steps. (1) The Morse and nullity indices of Hessian are characterized by a criterion developed by Neves (2009). (2) Constraints are used to connect to the sign of $Q'(\omega)$.

Work Flow

Our goal is to show energetic stability.

- (Done) Existence of Periodic Waves
- (Done) Construct period function $T(\mathcal{E}, \omega)$ and analyze to establish monotonicity.
- (Done) Use monotonicity to characterize the number of negative eigenvalues of Hessian operator.
- (Done) Impose constraint(s) to remove negative directions and leads us to energetic stability.
- This gives us a criterion of stability using monotonicity of frequency-mass mapping $\omega \mapsto Q$:
  - (Even waves) increasing: minimizer in $H^1_{\rm per}$; decreasing: $H^1_{\rm per}$ saddle point.
  - (Odd waves) increasing: minimizer in $\mathcal{Y} \subset H^1_{\rm per}$; decreasing: $H^1_{\rm per}$ saddle point.

  Can we analyze monotonicity of curve $Q(\omega)$?

Monotonicity of Mass

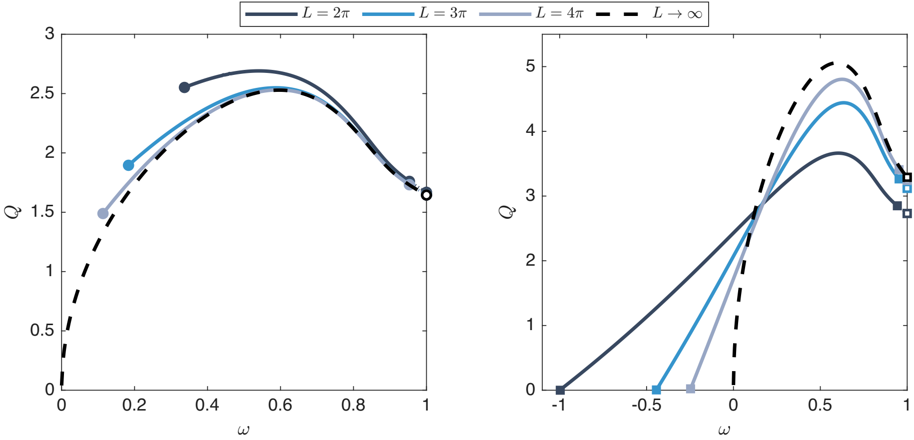

Figure: Mass-frequency curve $Q(\omega)$: (Left) even wave; (right) odd wave.

Monotonicity of Mass

We recently proved endpoint monotonicity of $Q(\omega)$.

Theorem

For $L > 0$, define $\omega_L = \frac{2\pi^2}{L^2+2\pi^2}$ and $\Omega_L = -\frac{4\pi^2}{L^2}$ and the mass functional

$$Q = -\int_{\mathbb{T}_L} \log\left(1-\phi^2\right) dx$$

- *is monotonically increasing near $\omega = \omega_L^+$ ($\omega = \Omega_L^+$) for even (odd) waves where the even waves require a critical length $L > \frac{2\pi}{3^{1/4}}$.*
- *is monotonically decreasing near $\omega = 1^-$ for both even and odd waves.*

Remark The proof is an explicit analysis.

- The left endpoint case is a straightforward corollary using series expansion.
- The right endpoint case requires construction of a technical lemma to evaluate left derivatives at $\omega = 1^-$, and then estimate sign-definite integrals.

Monotonicity of Mass

We recently proved endpoint monotonicity of $Q(\omega)$.

Theorem

For $L > 0$, define $\omega_L = \frac{2\pi^2}{L^2+2\pi^2}$ and $\Omega_L = -\frac{4\pi^2}{L^2}$ and the mass functional

$$Q = -\int_{\mathbb{T}_L} \log\left(1 - \phi^2\right) dx$$

- *is monotonically increasing near $\omega = \omega_L^+$ $(\omega = \Omega_L^+)$ for even (odd) waves where the even waves require a critical length $L > \frac{2\pi}{3^{1/4}}$.*
- *is monotonically decreasing near $\omega = 1^-$ for both even and odd waves.*

Remark The proof is an explicit analysis.

- The left endpoint case is a straightforward corollary using series expansion.
- The right endpoint case requires construction of a technical lemma to evaluate left derivatives at $\omega = 1^-$, and then estimate sign-definite integrals.

Monotonicity of Mass

We recently proved endpoint monotonicity of $Q(\omega)$.

Theorem

For $L > 0$, define $\omega_L = \frac{2\pi^2}{L^2+2\pi^2}$ and $\Omega_L = -\frac{4\pi^2}{L^2}$ and the mass functional

$$Q = -\int_{\mathbb{T}_L} \log\left(1 - \phi^2\right) dx$$

- *is monotonically increasing near $\omega = \omega_L^+$ ($\omega = \Omega_L^+$) for even (odd) waves where the even waves require a critical length $L > \frac{2\pi}{3^{1/4}}$.*
- *is monotonically decreasing near $\omega = 1^-$ for both even and odd waves.*

Remark The proof is an explicit analysis.

- The left endpoint case is a straightforward corollary using series expansion.
- The right endpoint case requires construction of a technical lemma to evaluate left derivatives at $\omega = 1^-$, and then estimate sign-definite integrals.

Summary

$$\text{(NLS-IDD)} \quad iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{T}_L.$$

Main Results

- Local energetic stability of standing periodic waves with the smooth profiles.
- Endpoint monotonicity of mapping $\omega \mapsto Q$

Future Works & Open Problems

- Convexity of mapping $\omega \mapsto Q$.
- Well-posedness of NLS-IDD in $H^1(\mathbb{R})$.

Paper

Fábio Natali, Dmitry E. Pelinovsky, and Shuoyang Wang,
Stability of periodic waves in the model with intensity-dependent dispersion,
Nonlinearity submitted, arXiv preprint arXiv:2603.28514, 2026 (38 pages)

Thank you! Questions?

Summary

$$\boxed{\text{(NLS-IDD)} \quad iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{T}_L.}$$

Main Results

- Local energetic stability of standing periodic waves with the smooth profiles.
- Endpoint monotonicity of mapping $\omega \mapsto Q$

Future Works & Open Problems

- Convexity of mapping $\omega \mapsto Q$.
- Well-posedness of NLS-IDD in $H^1(\mathbb{R})$.

Paper

Fábio Natali, Dmitry E. Pelinovsky, and Shuoyang Wang,
Stability of periodic waves in the model with intensity-dependent dispersion,
Nonlinearity submitted, arXiv preprint arXiv:2603.28514, 2026 (38 pages)

Thank you! Questions?

Summary

$$\boxed{\text{(NLS-IDD)} \quad iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{T}_L.}$$

Main Results

- Local energetic stability of standing periodic waves with the smooth profiles.
- Endpoint monotonicity of mapping $\omega \mapsto Q$

Future Works & Open Problems

- Convexity of mapping $\omega \mapsto Q$.
- Well-posedness of NLS-IDD in $H^1(\mathbb{R})$.

Paper

Fábio Natali, Dmitry E. Pelinovsky, and Shuoyang Wang,
Stability of periodic waves in the model with intensity-dependent dispersion,
Nonlinearity submitted, arXiv preprint arXiv:2603.28514, 2026 (38 pages)

Thank you! Questions?

Summary

$$\text{(NLS-IDD)} \quad iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{T}_L.$$

Main Results

- Local energetic stability of standing periodic waves with the smooth profiles.
- Endpoint monotonicity of mapping $\omega \mapsto Q$

Future Works & Open Problems

- Convexity of mapping $\omega \mapsto Q$.
- Well-posedness of NLS-IDD in $H^1(\mathbb{R})$.

Paper

Fábio Natali, Dmitry E. Pelinovsky, and Shuoyang Wang,
Stability of periodic waves in the model with intensity-dependent dispersion,
Nonlinearity submitted, arXiv preprint arXiv:2603.28514, 2026 (38 pages)

Thank you! Questions?