Periodic Waves of Nonlinear Schrödinger Equation with Intensity-Dependent Dispersion

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This thesis work is written as a research paper [34] and has been submitted to peer-reviewed journal prior to the writing of this thesis. Hence, the main results in the thesis are identical to [34] with a different organization and presentation, and the only additional material is the endpoint monotonicity analysis Section 4.

Abstract

We study standing periodic waves modeled by the nonlinear Schrödinger equation with the intensity-dependent dispersion coefficient. Spatial periodic profiles are smooth if the frequency of the standing waves is below the limiting frequency, for which the profiles become peaked (piecewisely continuous differentiable with a finite jump of the first derivative). We prove that there exist two families of the periodic waves with smooth profiles separated by a homoclinic orbit and the period function (the energy-to-period mapping) is monotonically increasing for the family inside the homoclinic orbit and decreasing for the family outside the homoclinic orbit. This property justifies the sharp criterion for the energetic stability of such standing periodic waves in the time evolution when the perturbations are restricted to be periodic with the same period for both families and, additionally for the family outside the homoclinic orbit, spatially odd with respect to the half-period. By approximating the sharp stability criterion numerically, we show that both families are energetically stable for small frequencies but become unstable when the frequency approaches the limiting frequency.

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1 Introduction

We consider the nonlinear Schrödinger (NLS) equation, where the dispersion coefficient depends linearly on the wave intensity. This model in one spatial dimension can be written in the normalized form:

$$iu_t + (1 - |u|^2)u_{xx} + |u|^2 u = 0, \tag{1.1}$$

where $u = u(t, x)$ and $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. We assume that $u(t, \cdot)$ is spatially periodic with the period L for any $t \in \mathbb{R}$. If the dispersion coefficient is constant, the model is equivalent to the cubic focusing NLS equation, one of the fundamental models of the nonlinear science [10, 17]. We refer to the model (1.1) as *the NLS–IDD equation*.

1.1 Background and motivations

Mathematical models with the intensity-dependent dispersion terms have been considered in the physics of the coherently prepared multistate atoms [13], quantum well waveguides [19], fiber-optics communication systems [23], and the quantum harmonic oscillators in the presence of nonlinear effective masses [4].

The NLS–IDD equation also arises in the continuum limit of the Salerno lattice model [33],

$$i\partial_\tau \psi_n + (1 - |\psi_n|^2)(\psi_{n+1} + \psi_{n-1}) + \mu|\psi_n|^2\psi_n = 0, \tag{1.2}$$

where $\mu \in \mathbb{R}$ is the coefficient of the onsite nonlinearity and $\psi_n = \psi_n(\tau)$ is the wave function in $(\tau, n) \in \mathbb{R} \times \mathbb{Z}$. If $\mu = 2 + h^2$ and $\psi_n(\tau) = e^{2i\tau}u(h^2\tau, hn)$ with a smooth $u = u(t, x)$, then expanding in powers of the small stepsize h yields the NLS–IDD equation (1.1) from the Salerno model (1.2) at the order of $\mathcal{O}(h^2)$.

The mathematical analysis of the model (1.1) without the local cubic term $|u|^2u$ was developed in [32], where it was shown that a continuous family of bright solitons exists among the standing wave solutions. The spatial profiles of bright solitons are singular with two logarithmic singularities for the first derivative and the continuous parameter is given by the distance between the two singularities. The energetic stability of the solitary waves was obtained in [30] by using the variational characterization of the singular profiles as minimizers of the mass subject to a fixed energy. Well-posedness of the model was not studied in [30, 32].

A similar model without the local cubic term $|u|^2u$ and with the inverted intensity–dependent coefficient $(1 - |u|^2)^{-1}u_{xx}$ was considered in [28], where a family of dark solitons (traveling wave solutions) was shown to have smooth spatial profiles and the limiting black solitons (standing wave solutions) were shown to be energetically stable as constrained minimizers of the energy subject to fixed mass and momentum. Dark solitons in the quasilinear NLS equations with nonconstant dispersion terms were considered in [20, 21, 22]. Both bright and dark solitons were also studied in the NLS equations with regularized dispersion terms [1, 2, 29].

The NLS–IDD equation (1.1) was studied in [18], where the continuous family of bright solitons is parameterized by the frequency of the standing wave solution $u(t, x) = e^{i\omega t}\phi(x)$ with the spatially decaying profile ϕ. The profiles of solitary waves are smooth for $0 < \omega < 1$

and peaked (piecewise continuous with a single jump of the first derivative) for $\omega = 1$. A sharp criterion for energetic stability of bright solitons in $H^1(\mathbb{R})$ was obtained for $0 < \omega < 1$ in [18] from the variational characterization of the smooth profiles as local minimizers of the energy subject to a fixed mass. The sharp criterion is given by the monotone increase of the mass with respect to the frequency, the latter condition is checked numerically.

The energetic stability is equivalent to the orbital stability if the local well-posedness of the NLS-IDD equation (1.1) can be obtained in $H^1(\mathbb{R})$. However, the state-of-the-art in the well-posedness of quasilinear NLS equations is not yet at the level of $H^1(\mathbb{R})$. Local well-posedness of the models which include (1.1) was proven in Sobolev spaces of higher regularity [15, 25, 31]. More recently, the local well-posedness of quasilinear NLS equations was established in $H^s(\mathbb{R})$ for $s > 2$ in [26] and for small data in $H^s(\mathbb{R})$ for $s > 1$ in [14]. Local well-posedness of quasilinear NLS equations including the NLS–IDD equation (1.1) was also extended to the periodic domain in Sobolev spaces H^s_{per} of higher regularity [7, 8, 9].

The main purpose of this work is to consider the energetic stability of standing periodic waves with the smooth profiles with respect to periodic perturbations of the same period. The periodic spatial domain is more practical for physical experiments modeled by the NLS–IDD equation (1.1). The mathematical analysis of stability in the periodic setting introduces additional challenges because the Morse index with a precise analysis of monotonicity of period function (the energy-to-period mapping). Similarly to the scopes of [18], we obtain a sharp criterion for the energetic stability of the smooth periodic waves as local minimizers of the energy in H^1_{per} subject to fixed mass, provided that the mass at the periodic wave profile is monotonically increasing with respect to frequency ω for $0 < \omega < 1$. We compute the latter criterion numerically, where we have discovered inaccuracies in the previous numerical approximations in [18] performed for the case of bright solitons. These main results of our study are described next.

1.2 Summary of Main Results

We denote the space of all square integrable functions which are L-periodic functions by L^2_{per}. For $s \geq 0$, the Sobolev space H^s_{per} is the set of all periodic distributions such that

$$\|f\|_{H^s_{\text{per}}} := \left(\sum_{k=-\infty}^{\infty} (1 + |k|^2)^s |\hat{f}(k)|^2 \right)^{1/2} < \infty,$$

where $\hat{f}$ is the periodic Fourier transform of f (the Fourier series of f). The space H^s_{per} is a Hilbert space with natural inner product denoted by $(\cdot,\cdot)_{H^s}$. When $s = 0$, the space H^s_{per} is isometrically isomorphic to the space L^2_{per}, that is, $L^2_{\text{per}} = H^0_{\text{per}}$. The norm and inner product in L^2_{per} are denoted by $\|\cdot\|_{L^2_{\text{per}}}$ and $\langle \cdot,\cdot \rangle_{L^2{}_{\text{per}}}$.

The time-dependent NLS–IDD equation (1.1) admits the conserved energy $H(u)$ and mass $Q(u)$ given by

$$H(u) = \int_{\mathbb{T}_L} \left(|u_x|^2 + |u|^2 + \log(1 - |u|^2) \right) dx \tag{1.3}$$

and

$$Q(u) = -\int_{\mathbb{T}_L} \log(1-|u|^2)dx. \tag{1.4}$$

where $\int_{\mathbb{T}_L}$ denotes the integral over the periodic domain $\mathbb{T}_L$ with the spatial period L, which is independent on the starting point of integration. The conserved quantities are well defined in the set of functions

$$\mathcal{X} = \left\{ u \in H^1_{\rm per} : \quad \|u\|_{L^\infty} < 1 \right\}.$$

The NLS–IDD equation (1.1) also admits the conserved momentum $P(u)$ if $u \neq 0$, see [18]. Since the momentum does not play any role in our study, we do not introduce it here.

We consider standing waves of the form $u(t,x) = e^{i\omega t}\phi(x)$, where ω is the wave frequency. Substituting this ansatz into (1.1), we obtain

$$-(1-\phi^2)\phi'' + \omega\phi - \phi^3 = 0, \tag{1.5}$$

which can be rewritten as Newton's equation for a 1D particle in a potential energy V:

$$\frac{d^2\phi}{dx^2} = \frac{(\omega-\phi^2)}{1-\phi^2}\phi = -\frac{dV}{d\phi}, \qquad V(\phi) = \frac{1}{2}(\omega-\phi^2) + \frac{1}{2}(1-\omega)\log\frac{1-\omega}{1-\phi^2}. \tag{1.6}$$

The total energy E of Newton's particle is conserved along every solution of (1.6):

$$E(\phi,\phi') = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi). \tag{1.7}$$

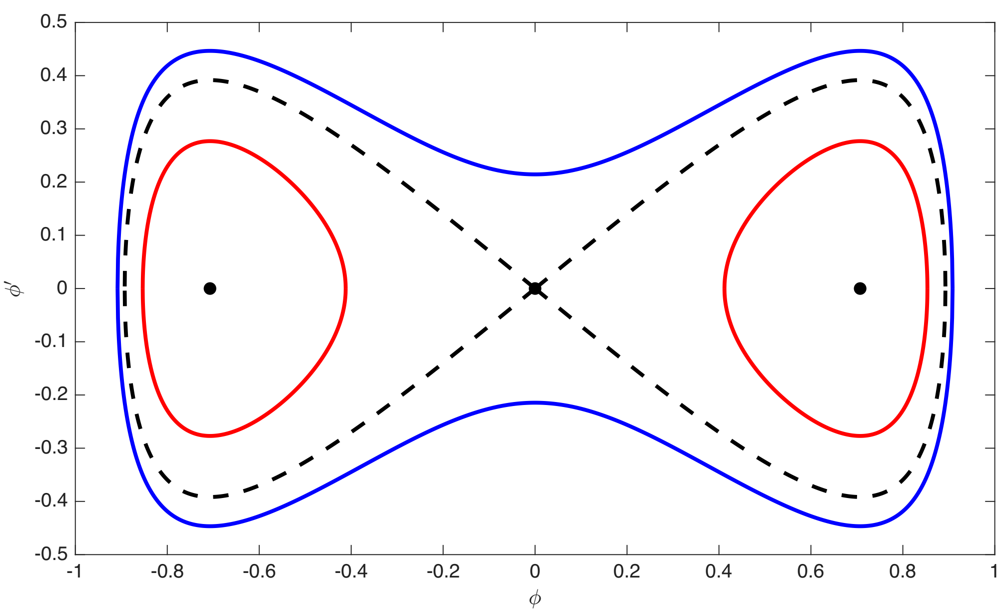

Figure 1.1: The phase portrait of system (1.6) for $\omega = 0.5$.

The variational characterization of the spatial profile ϕ is possible since the second-order equation (1.6) is the Euler–Lagrange equation for the augmented energy functional

$$G(u) = H(u) + \omega Q(u), \tag{1.8}$$

defined from the conserved energy $H(u)$ and mass $Q(u)$ in (1.3) and (1.4).

The phase portrait in Figure 1.1 represents all bounded solutions of the system (1.6) for $0 < \omega < 1$, see also [18]. There exist two families of the periodic orbits with smooth profiles separated by a pair of homoclinic orbits. One family is inside one of the two homoclinic orbits with the left (negative) periodic orbits being symmetrically reflected from the right (positive) periodic orbits due to the symmetry transformation: $\phi \to -\phi$. The other family is outside the two homoclinic orbits and spans symmetrically all four quadrants of the phase plane.

The following theorem summarizes the existence properties of the two families of the periodic orbits, and it is formally proved in [34].

Theorem 1.1. *Fix the spatial period $L > 0$ for the periodic domain $\mathbb{T}_L$ and define*

$$\omega_L = \frac{2\pi^2}{L^2 + 2\pi^2}, \quad \Omega_L = -\frac{4\pi^2}{L^2}.$$

For any $\omega \in (\omega_L, 1)$, there exists a periodic orbit of system (1.6) with the smooth profile ϕ satisfying

$$\begin{cases} 0 < \phi(x) < 1, & & \forall x \in \mathbb{T}_L, \\ \phi(x - x_0) = \phi(x_0 - x), & x_0 \in \mathbb{T}_L, & \forall x \in \mathbb{T}_L. \end{cases} \tag{1.9}$$

For any $\omega \in (\Omega_L, 1)$, there exists a periodic orbit of system (1.6) with the smooth profile ϕ satisfying

$$\begin{cases} -1 < \phi(x) < 1, & & \forall x \in \mathbb{T}_L, \\ \phi(x - x_0) = -\phi(x_0 - x) = \phi\left(\frac{L}{2} - x + x_0\right), & x_0 \in \mathbb{T}_L, & \forall x \in \mathbb{T}_L. \end{cases} \tag{1.10}$$

For both families, x_0 is an arbitrary translational parameter along the periodic orbit.

Remark 1.2. *For simplicity of terminology, we call the family of periodic orbits inside the homoclinic orbits satisfying (1.9) as the even waves and the family of periodic orbits outside the homoclinic orbits satisfying (1.10) as the odd waves. Figure 1.1 shows a former periodic orbit in blue and a latter periodic orbit in red together with its symmetric reflection. The homoclinic orbits are shown by dashed black lines.*

Each family of periodic orbits corresponds to the energy level $E(\phi, \phi') = \mathcal{E}$ in the first invariant (1.7). For $\omega \in (0, 1)$, the family of even waves (1.9) corresponds to $\mathcal{E} \in (0, \mathcal{E}_\omega)$ and the family of odd waves (1.10) corresponds to $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$, where

$$\mathcal{E}_\omega = \frac{1}{2}\omega + \frac{1}{2}(1 - \omega)\log(1 - \omega)$$

is the energy level corresponding to the homoclinic orbits for the saddle point $(0, 0)$. If $\omega \in (-\infty, 0)$, the family of odd waves satisfying (1.10) correspond to $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$, where $\mathcal{E}_\omega = V(0)$ is the energy level corresponding to the center point (0,0). For each energy level $E(\phi, \phi') = \mathcal{E}$, we can define the period function $T(\mathcal{E}, \omega)$ by

$$T(\mathcal{E}, \omega) = \oint \frac{d\phi}{\sqrt{2(\mathcal{E} - V(\phi))}}, \tag{1.11}$$

where $\oint$ corresponds to the line integral taken along the closed periodic orbit. Figure 1.2 shows the dependence of $T(\mathcal{E}, \omega)$ versus $\mathcal{E}$ for fixed values of $\omega \in (0, 1)$, where the divergence of $T(\mathcal{E}, \omega)$ corresponds to the homoclinic orbit at $\mathcal{E} = \mathcal{E}_\omega$. The figure suggests that, for $\omega \in (0, 1)$, the mapping $\mathcal{E} \to T(\mathcal{E}, \omega)$ is monotonically increasing for the even wave and is monotonically decreasing for the odd wave. These properties are formulated in the following theorem.

Theorem 1.3. *The period function $T = T(\mathcal{E}, \omega)$ in (1.11) is C^1 function of $\mathcal{E} \in (0, \infty) \backslash \mathcal{E}_\omega$ if $\omega \in (0, 1)$ and $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$ if $\omega \in (-\infty, 1)$. For any $\omega \in (0, 1)$, the mapping $(0, \mathcal{E}_\omega) \ni \mathcal{E} \to T(\mathcal{E}, \omega)$ is monotonically increasing. For $\omega \in (-\infty, 1)$, the mapping $(\mathcal{E}_\omega, \infty) \ni \mathcal{E} \to T(\mathcal{E}, \omega)$ is monotonically decreasing.*

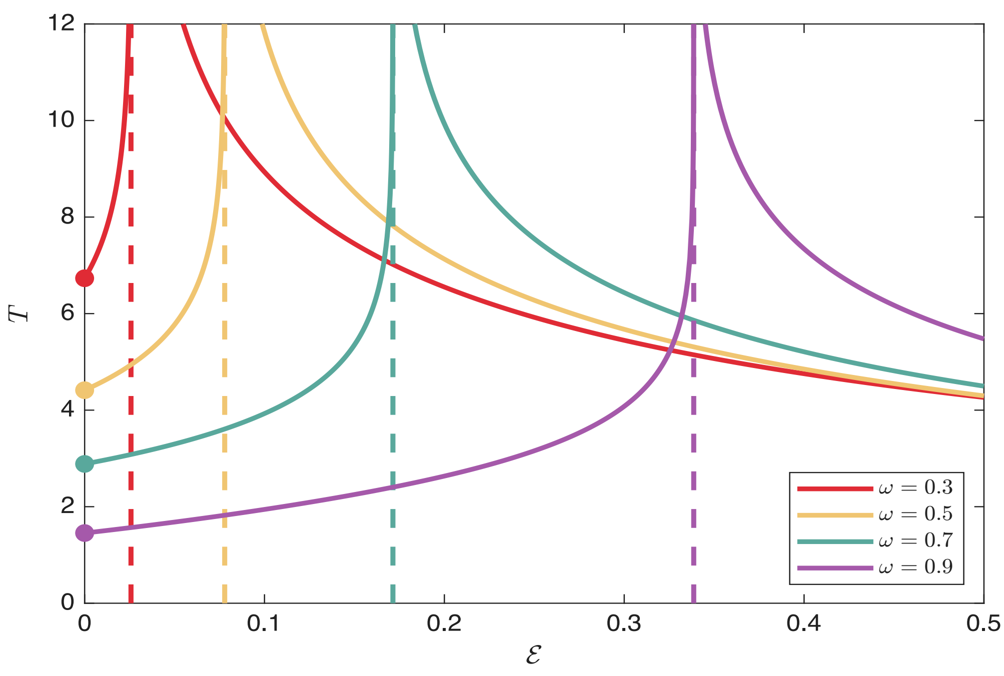

Figure 1.2: The period function $T(\mathcal{E}, \omega)$ versus $\mathcal{E}$ for fixed values of ω. The dots denote the cutoff value of $\mathcal{E}$ satisfying $T(\mathcal{E}, \omega) = \pi\sqrt{2(1-\omega)/\omega}$ for $\omega = \omega_L$. The vertical lines show divergence of $T(\mathcal{E}, \omega)$ at $\mathcal{E} = \mathcal{E}_\omega$.

Due to smoothness and monotonicity of the period function in Theorem 1.3, one can uniquely define the energy level $\mathcal{E} = \mathcal{E}_L(\omega)$ for any spatial period $L > 0$ in Theorem 1.1 from the root of $T(\mathcal{E}_L(\omega),\omega) = L$, where $\omega \in (\omega_L, 1)$ for the even wave and $\omega \in (\Omega_L, 1)$ for the odd wave. Furthermore, the mappings $(\omega_L, 1) \ni \omega \to \mathcal{E}_L(\omega)$ and $(\Omega_L, 1) \ni \omega \to \mathcal{E}_L(\omega)$ are C^1. These smoothness properties play a central role in the energetic stability analysis of the periodic waves.

The Hessian operator $\mathcal{L} = H''(\phi) + \omega Q''(\phi)$ of the augmented energy functional (1.8) at the critical point with the profile ϕ is defined as

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}, \qquad \begin{aligned} \mathcal{L}_+ &= -\partial_x^2 + 1 + (\omega - 1)\tfrac{1+\phi^2}{(1-\phi^2)^2}, \\ \mathcal{L}_- &= -\partial_x^2 + 1 + (\omega - 1)\tfrac{1}{(1-\phi^2)}, \end{aligned} \tag{1.12}$$

To simplify notation, we set

$$\mathbb{H}^s_{\text{per}} := H^s_{\text{per}} \times H^s_{\text{per}}, \quad \mathbb{L}^2_{\text{per}} := L^2_{\text{per}} \times L^2_{\text{per}},$$

endowed with their usual norms and scalar products. When necessary and since $\mathbb{C}$ can be identified with $\mathbb{R}^2$, notations above can also be used in the complex-valued functions in the following sense: for $f \in \mathbb{H}^s_{\text{per}}$ we have $f = f_1 + if_2 \equiv (f_1, f_2)$, where $f_1, f_2 \in H^s_{\text{per}}$.

By studying the spectrum of $\mathcal{L}$ in $\mathbb{L}^2_{\text{per}}$, we obtain the sharp criterion for the energetic stability of the periodic waves with the spatial profile ϕ stated in the following theorem.

Theorem 1.4. *Fix the spatial period $L > 0$ as in Theorem 1.1 and set $x_0 = 0$. The profile $\phi \in H^1_{\text{per}}$ is a C^1 function of ω for the even wave in $(\omega_L, 1)$ and for the odd wave in $(\Omega_L, 1)$. For any $\omega \in (\omega_L, 1)$, the even wave with the profile ϕ is a local minimizer of energy $H(u)$ for a fixed mass $Q(u)$ in H^1_{per}, which is degenerate only due to translational and rotational symmetries, if and only if the mapping $\omega \to Q(\phi)$ is monotonically increasing. For any $\omega \in (\Omega_L, 1)$, the odd wave with the profile ϕ satisfying a local minimizer of energy $H(u)$ for a fixed mass $Q(u)$ in $\mathcal{Y} \subset H^1_{\text{per}}$, where*

$$\mathcal{Y} = \left\{ u \in H^1_{\text{per}} : \quad u\left(\frac{L}{2} - x\right) = -u\left(x - \frac{L}{2}\right), \quad \forall x \in \mathbb{T}_L \right\}, \tag{1.13}$$

which is only degenerate by the rotational symmetry, if and only if the mapping $\omega \to Q(\phi)$ is monotonically increasing.

Based on the numerical approximations and the sharp criterion in Theorem 1.4, we conclude from Figure 5.3 that both even and odd periodic waves are energetically stable for smaller values of ω and energetically unstable for values of ω near $\omega = 1$. To be precise, we formulate the following conjecture.

Conjecture 1.5. *There is $\omega_* \in (\omega_L, 1)$ and $\Omega_* \in (0, 1)$ such that the even wave satisfying (1.9) is energetically stable for $\omega \in (\omega_L, \omega_*)$ and unstable for $\omega \in (\omega_*, 1)$, whereas the odd wave satisfying (1.10) is energetically stable for $\omega \in (\Omega_L, \Omega_*)$ and unstable for $\omega \in (\Omega_*, 1)$.*

2 Monotonicity of Period Functions

We prove Theorem 1.3 by analysing the period function $T = T(\mathcal{E}, \omega)$ introduced in (1.11). The period function is associated with the periodic orbits on the phase plane for the system of ordinary differential equations

$$\begin{cases} \phi' = \xi, \\ \xi' = \dfrac{\omega\phi - \phi^3}{1 - \phi^2}. \end{cases} \tag{2.1}$$

It follows from the standard theory of ordinary differential equations that the solution ϕ depends smoothly on the parameter $\mathcal{E} = E(\phi, \phi')$, where the energy function is

$$E(\phi, \phi') = \frac{1}{2}(\phi')^2 + V(\phi), \quad V(\phi) = \frac{1}{2}(\omega - \phi^2) + \frac{1}{2}(1 - \omega)\log\frac{1 - \omega}{1 - \phi^2}. \tag{2.2}$$

For $\omega \in (0, 1)$, the even wave satisfying (1.9) corresponds to $\mathcal{E} \in (0, \mathcal{E}_\omega)$ and the odd wave satisfying (1.10) corresponds to $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$, where $\mathcal{E}_\omega = E(0, 0)$ corresponds to the energy level of the pair of homoclinic orbits from the saddle point $(0, 0)$ of (2.1) which surround the center points $(\pm\sqrt{\omega}, 0)$ of (2.1). We note that

$$V(\pm\sqrt{\omega}) = 0 \quad \text{and} \quad \lim_{\phi \to \pm 1} V(\phi) = +\infty.$$

Furthermore, $V(\phi) \geq 0$ for all $\phi \in [-1, 1]$ and $V'(\phi) > 0$ for $\phi \in (\sqrt{\omega}, 1)$.

Section 2.1 gives the proof of $\partial_{\mathcal{E}} T(\mathcal{E}, \omega) > 0$, $\mathcal{E} \in (0, \mathcal{E}_\omega)$ for the periodic orbits inside the homoclinic orbit (the even waves). Section 2.2 gives the proof of $\partial_{\mathcal{E}} T(\mathcal{E}, \omega) < 0$, $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$ for the periodic orbits outside the pair of homoclinic orbits (the odd waves). The latter result also holds for $\omega \in (-\infty, 0)$ and $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$, for which $\mathcal{E}_\omega = E(0, 0)$ corresponds to the energy level of center point $(0, 0)$.

2.1 Monotonicity of Even Periodic Waves

By the main theorem in [5], the period function $T(\mathcal{E}, \omega)$ is monotonically increasing in $\mathcal{E}$ in $(0, \mathcal{E}_\omega)$ if $I''(\phi) > 0$ for $\phi \in (0, 1)$, where

$$I(\phi) = \frac{V(\phi)}{[V'(\phi)]^2}. \tag{2.3}$$

Note that the theorem in [5] can be applied because $V(\sqrt{\omega}) = 0$ is properly normalized at the centre point $(\sqrt{\omega}, 0)$. Computing

$$V'(\phi) = -\frac{\phi(\omega - \phi^2)}{1 - \phi^2}, \quad V''(\phi) = -\frac{\omega + (\omega - 3)\phi^2 + \phi^4}{(1 - \phi^2)^2}, \quad V'''(\phi) = \frac{2(1 - \omega)\phi(\phi^2 + 3)}{(1 - \phi^2)^3},$$

we obtain from (2.3) that

$$I'' = \frac{6V(V'')^2 - 2VV'V''' - 3(V')^2V''}{(V')^4} =: \frac{P}{(V')^4(1-\phi^2)^4}, \tag{2.4}$$

where

$$P(\phi) = 3\phi^2(\omega-\phi^2)^2A(\phi) + \left[3A(\phi)^2 + 2(1-\omega)\phi^2(3+\phi^2)(\omega-\phi^2)\right]B(\phi),$$

with

$$A(\phi) := \omega + (\omega-3)\phi^2 + \phi^4, \qquad B(\phi) := \omega - \phi^2 + (1-\omega)\log\frac{1-\omega}{1-\phi^2}.$$

Since P depends on ϕ^2, we introduce $t=\phi^2$ and redefine P, A, and B as a function of t:

$$P(t) = 3t(\omega-t)^2A(t) + \left[3A(t)^2 + 2(1-\omega)t(3+t)(\omega-t)\right]B(t), \quad t\in[0,1), \tag{2.5}$$

with

$$A(t) = \omega + (\omega-3)t + t^2, \qquad B(t) = \omega - t + (1-\omega)\log\frac{1-\omega}{1-t}. \tag{2.6}$$

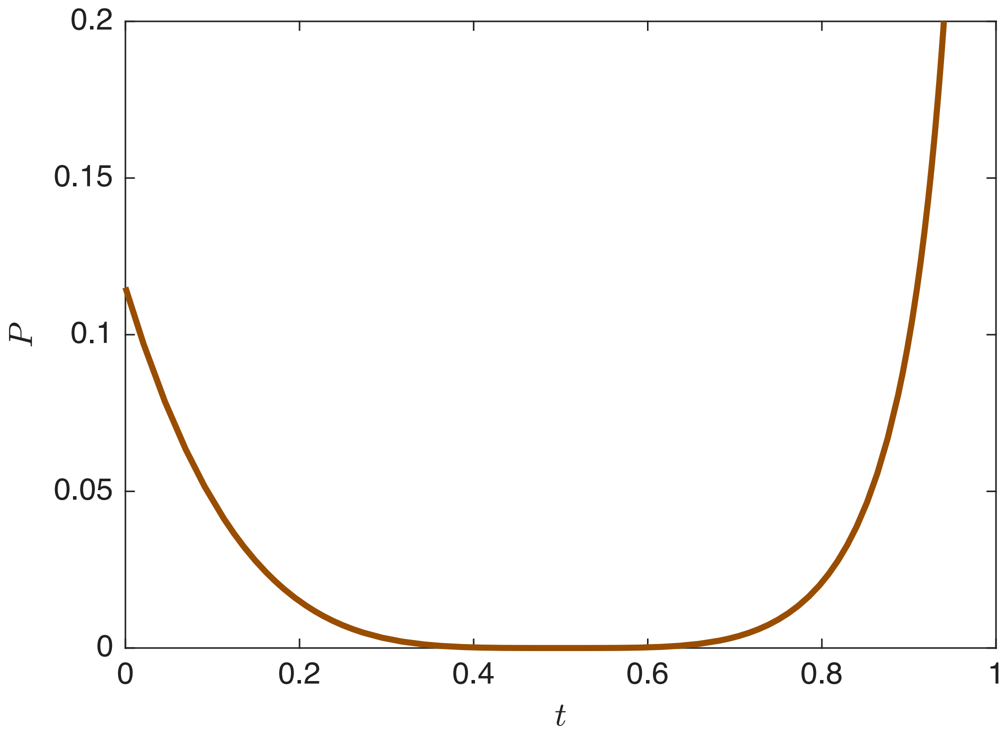

Figure 2.1: The dependence of P versus t given by (2.5) for $\omega = 0.5$.

Figure 2.1 shows the dependence of P versus t for $\omega = 0.5$. The plot suggests that

- $P(t)$ has a quadruple zero at $t = \omega$.
- $P(t) > 0$ for $t \in (0, \omega) \cup (\omega, 1)$.

These facts are proven rigorously in Lemmas 2.1 and 2.3 below.

Lemma 2.1. *The function P given by (2.5) with (2.6) is real analytic on $(0,1)$ and it admits a zero of the quadruple order at $t = \omega$, such that*

$$P'(\omega) = P''(\omega) = P'''(\omega) = 0, \qquad P^{(4)}(\omega) = \frac{4(9 - 6\omega - \omega^2)}{1 - \omega} > 0. \tag{2.7}$$

Consequently, there exists $\delta > 0$ such that $P(t) > 0$ for $t \in [\omega - \delta, \omega + \delta] \backslash \{\omega\}$.

Proof. The function $P(t)$ is real analytic on $(0,1)$, because the logarithmic function in B is analytic for $t < 1$ and other functions are polynomials in t. The Taylor series of P at $t = \omega$ can be written as

$$P(t) = \sum_{n=0}^{\infty} \frac{P^{(n)}(\omega)}{n!} (t - \omega)^n.$$

We have $B(\omega) = 0$, and

$$B'(t) = -\frac{\omega - t}{1 - t}, \quad B''(t) = \frac{1 - \omega}{(1 - t)^2}, \tag{2.8}$$

which implies

$$B(t) = \frac{(t - \omega)^2}{2(1 - \omega)} + \frac{(t - \omega)^3}{3(1 - \omega)^2} + \frac{(t - \omega)^4}{4(1 - \omega)^3} + \mathcal{O}((t - \omega)^5). \tag{2.9}$$

Furthermore, we define

$$G(t) := 3A(t)^2 + 2(1 - \omega)t(3 + t)(\omega - t) \tag{2.10}$$

and expand

$$\begin{aligned} G(t) &= 3\omega^2 - 12\omega t + (21 - 4\omega + \omega^2)t^2 - (20 - 8\omega)t^3 + 3t^4 \\ &= 12\omega^2(1 - \omega)^2 + 2\omega(1 - \omega)(15 - 19\omega)(t - \omega) \\ &\quad + (1 - \omega)(21 - 43\omega)(t - \omega)^2 + \mathcal{O}((t - \omega)^3). \end{aligned} \tag{2.11}$$

Similarly, we expand

$$\begin{aligned} 3tA(t) &= 3t(\omega + (\omega - 3)t + t^2) \\ &= -6\omega^2(1 - \omega) - 15\omega(1 - \omega)(t - \omega) - (9 - 12\omega)(t - \omega)^2 + \mathcal{O}((t - \omega)^3). \end{aligned} \tag{2.12}$$

Substituting (2.9) into (2.5) yields

$$P(t) = (t-\omega)^2 \left[3tA(t) + G(t) \left(\frac{1}{2(1-\omega)} + \frac{(t-\omega)}{3(1-\omega)^2} + \frac{(t-\omega)^2}{4(1-\omega)^3} + \mathcal{O}((t-\omega)^3) \right) \right].$$

By using (2.11) and (2.12), we compute coefficients of powers $(t-\omega)$ in $P(t)$:

$$\begin{aligned}
(t-\omega)^2: \quad & -6\omega^2(1-\omega) + \frac{12\omega^2(1-\omega)^2}{2(1-\omega)} = 0, \\
(t-\omega)^3: \quad & -15\omega(1-\omega) + \frac{2\omega(1-\omega)(15-19\omega)}{2(1-\omega)} + \frac{12\omega^2(1-\omega)^2}{3(1-\omega)^2} = 0, \\
(t-\omega)^4: \quad & -(9-12\omega) + \frac{(1-\omega)(21-43\omega)}{2(1-\omega)} + \frac{2\omega(1-\omega)(15-19\omega)}{3(1-\omega)^2} + \frac{12\omega^2(1-\omega)^2}{4(1-\omega)^3} \\
& = \frac{9-6\omega-\omega^2}{6(1-\omega)},
\end{aligned}$$

This yields (2.7).

The integral remainder of $P(t)$ can be written in the integral form

$$P(t) = \frac{1}{3!}(t-\omega)^4 \int_0^1 (1-s)^3 P^{(4)}(\omega + s(t-\omega))\, ds.$$

and there exists $\delta' > 0$, such that $P^{(4)}(t)$ is continuous on $t \in (\omega - \delta', \omega + \delta')$. By taking $\epsilon = \frac{1}{2}P^{(4)}(\omega)$ and $\delta < \delta'$, there is a local strictly positive estimation

$$P(t) \geq \frac{1}{48} P^{(4)}(\omega)(t-\omega)^4 > 0, \qquad t \in [\omega - \delta, \omega) \cup (\omega, \omega + \delta],$$

which yields the assertion on positivity of $P(t)$ near $t = \omega$. □

To estimate the global behavior of the functon $P(t)$ for $t \in (0,1)$, we use the following bounds on the function $B(t)$ obtained from (2.8).

Lemma 2.2. *The function B defined in (2.6) can be estimated as*

$$\frac{(\omega-t)^2}{2(1-t)} \leq B(t) \leq \frac{(\omega-t)^2}{2(1-\omega)}, \qquad t \in (0,\omega), \tag{2.13}$$

and

$$B(t) \leq \frac{(\omega-t)^2}{2(1-t)}, \qquad t \in (\omega, 1). \tag{2.14}$$

Proof. It follows from (2.8) that B can be written in the integral form:

$$B(t) = \int_t^\omega \frac{\omega - s}{1-s}\, ds, \quad t \in (0,1).$$

For $t \in (0, \omega)$, let $0 < t \leq s \leq \omega$, so that $\frac{1}{1-t} \leq \frac{1}{1-s} \leq \frac{1}{1-\omega}$. Then, we have

$$B(t) \leq \frac{1}{1-\omega} \int_t^\omega (\omega - s)\, ds = \frac{(\omega - t)^2}{2(1-\omega)}$$

and

$$B(t) \geq \frac{1}{1-t} \int_t^\omega (\omega - s)\, ds = \frac{(\omega - t)^2}{2(1-t)}.$$

This yields (2.13). Similarly, for $t \in (\omega, 1)$, let $\omega \leq s \leq t < 1$, so that $\frac{1}{1-t} \geq \frac{1}{1-s}$. Then, we have

$$B(t) \leq \frac{1}{1-t} \int_t^\omega (\omega - s)\, ds = \frac{(\omega - t)^2}{2(1-t)}.$$

This yields (2.14). $\square$

We use Lemma 2.2 to extend Lemma 2.1 and to guarantee that $P(t)$ is positive for every $t \in (0, 1)$. This is obtained by controlling the derivative of P in t separately for $t \in (0, \omega)$ and $t \in (\omega, 1)$.

Lemma 2.3. *The function P is monotonically decreasing on $(0, \omega)$ and increasing on $(\omega, 1)$.*

Proof. To show that $P'(t) < 0$ for $t \in (0, \omega)$ and $P'(t) > 0$ for $t \in (\omega, 1)$, we use (2.5) rewritten as

$$P(t) = Q(t) + B(t)G(t),$$

where $Q(t) := 3t(\omega - t)^2 A(t)$ with A and B defined in (2.6) and G defined in (2.10). By using (2.8) for $B'(t)$, as well as

$$\begin{aligned}
Q'(t) &= 3(t - \omega)\left[5t^3 + (\omega - 12)t^2 + \omega(9 - 2\omega)t - \omega^2\right], \\
G'(t) &= 2\left[6t^3 + 6(2\omega - 5)t^2 + (\omega^2 - 4\omega + 21)t - 6\omega\right],
\end{aligned}$$

we obtain

$$\begin{aligned}
P'(t) &= Q'(t) + B'(t)G(t) + B(t)G'(t) \\
&= \left[B(t) - \frac{(\omega - t)^2}{2(1-t)}\right] G'(t) + \frac{(\omega - t)^2}{1-t}\left[\frac{1}{2}G'(t) + \frac{1-t}{(\omega - t)^2}(Q'(t) + B'(t)G(t))\right] \\
&= \left[B(t) - \frac{(\omega - t)^2}{2(1-t)}\right] G'(t) - \frac{(\omega - t)^3}{1-t}\left[6 + (1-\omega)t - 6t^2\right], \qquad (2.15)
\end{aligned}$$

where the last identity is derived directly from

$$\begin{aligned}
&\frac{1-t}{(\omega - t)^2}(Q'(t) + B'(t)G(t)) \\
&= \frac{1}{t - \omega}\left[3(1-t)\left[5t^3 + (\omega - 12)t^2 + \omega(9 - 2\omega)t - \omega^2\right] + G(t)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{t-\omega}\left[-12t^4 + (31+5\omega)t^3 + (7\omega^2 - 28\omega - 15)t^2 + 3\omega(5-\omega)t\right] \\
&= -12t^3 + (31-7\omega)t^2 + 3(\omega-5)t
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}G'(t) + \frac{1-t}{(\omega-t)^2}(Q'(t) + B'(t)G(t)) &= -6t^3 + (5\omega+1)t^2 + (\omega^2 - \omega + 6)t - 6\omega \\
&= (\omega - t)\left[6t^2 + (\omega-1)t - 6\right].
\end{aligned}
$$

For $t \in (0, \omega)$, we use the estimate (2.13) and obtain

$$
0 \le B(t) - \frac{(\omega-t)^2}{2(1-t)} \le \frac{(\omega-t)^2}{2(1-\omega)} - \frac{(\omega-t)^2}{2(1-t)} = \frac{(\omega-t)^3}{2(1-t)(1-\omega)}. \tag{2.16}
$$

Since

$$
6 + (1-\omega)t - 6t^2 \ge \min\{6, 1-\omega\} > 0, \quad t \in [0,1],
$$

for every $\omega \in (0,1)$, it follows from (2.15) and the lower bound in (2.16) that $P'(t) < 0$ for $t \in (0,\omega)$ if $G'(t) < 0$. On the other hand, if $G'(t) > 0$, then we use the upper bound in (2.16) and obtain

$$
\begin{aligned}
P'(t) &\le \frac{(\omega-t)^3}{(1-t)(1-\omega)}\left[\frac{1}{2}G'(t) - (1-\omega)(6 + t - \omega t - 6t^2)\right] \\
&= \frac{2(\omega-t)^3}{(1-t)(1-\omega)}\left[3t^3 + 3(\omega-4)t^2 + (10-\omega)t - 3\right].
\end{aligned}
$$

We show that the last expression in the brackets is negative, which yields $P'(t) < 0$ for $t \in (0,\omega)$ if $G'(t) > 0$. Indeed, we have

$$
3t^3 + 3(\omega-4)t^2 + (10-\omega)t - 3 = 3(t-1)^3 - (1-\omega)t(3t-1)
$$

which implies

$$
3(t-1)^3 - (1-\omega)t(3t-1) \le 3(t-1)^3 < 0, \quad \frac{1}{3} \le t < 1
$$

and

$$
3(t-1)^3 - (1-\omega)t(3t-1) = (t-1)\left[3(t-1)^2 + \frac{1-\omega}{1-t}t(3t-1)\right] < 0, \quad 0 < t \le \frac{1}{3},
$$

since $\frac{1-\omega}{1-t} < 1$ for $t \in (0,\omega)$ and

$$
3(t-1)^2 + \frac{1-\omega}{1-t}t(3t-1) \ge 3(t-1)^2 + t(3t-1) = 3 - 7t + 6t^2 \ge \frac{4}{3}, \qquad 0 \le t \le \frac{1}{3}.
$$

For $t \in (\omega, 1)$, we use again that

$$\frac{1}{2}G'(t) - (1-\omega)(6+(1-\omega)t-6t^2) = 2\left[3(t-1)^3 - (1-\omega)t(3t-1)\right] < 0,$$

which yields

$$\frac{G'(t)}{6+(1-\omega)t-6t^2} < 2(1-\omega).$$

By using (2.14), we know that $B(t) - \frac{(\omega-t)^2}{2(1-t)} \leq 0$, so that we can estimate (2.15) for $t \in (\omega, 1)$ as follows:

$$\begin{aligned}
P'(t) &= (6+(1-\omega)t-6t^2)\left\{\left[B(t) - \frac{(\omega-t)^2}{2(1-t)}\right]\frac{G'(t)}{6+(1-\omega)t-6t^2} - \frac{(\omega-t)^3}{1-t}\right\} \\
&> (6+(1-\omega)t-6t^2)\left\{2(1-\omega)\left[B(t) - \frac{(\omega-t)^2}{2(1-t)}\right] - \frac{(\omega-t)^3}{1-t}\right\} \\
&= (6+(1-\omega)t-6t^2)\left\{2(1-\omega)B(t) - (t-\omega)^2\right\}.
\end{aligned}$$

By using the definition of B in (2.6) and the variable $x := \frac{t-\omega}{1-\omega} \in (0,1)$, we get Taylor series expansion

$$\begin{aligned}
2(1-\omega)B(t) - (t-\omega)^2 &= 2(1-\omega)^2 \log\frac{1-\omega}{1-t} - 2(1-\omega)(t-\omega) - (t-\omega)^2 \\
&= 2(1-\omega)^2\left[-\log(1-x) - x - \frac{1}{2}x^2\right] \\
&= 2(1-\omega)^2 \sum_{n=3}^{\infty}\frac{x^n}{n},
\end{aligned}$$

which is strictly positive for $x \in (0,1)$. Hence, $P'(t) > 0$ for $t \in (\omega, 1)$. $\square$

The period function $T(\mathcal{E}, \omega)$ given by (1.11) can be rewritten for the even periodic waves explicitly by

$$T_{\text{even}}(\mathcal{E},\omega) = 2\int_m^M \frac{d\phi}{\sqrt{2\mathcal{E} + (1-\omega)\log(1-\phi^2) - (1-\omega)\log(1-\omega) + \phi^2 - \omega}}, \tag{2.17}$$

where

$$m := \min_{x\in[-\frac{L}{2},\frac{L}{2}]} \phi(x) \in (0, \sqrt{\omega}) \quad \text{and} \quad M := \max_{x\in[-\frac{L}{2},\frac{L}{2}]} \phi(x) \in (\sqrt{\omega}, 1) \tag{2.18}$$

are given by roots of $V(\phi) = \mathcal{E}$ for $\mathcal{E} \in (0, \mathcal{E}_\omega)$. By using Lemma 2.3, we prove monotonicity of the period function in $\mathcal{E}$ stated in Theorem 1.3.

Proposition 2.4. *For every $\omega \in (0,1)$, the period function $T_{\text{even}}(\mathcal{E},\omega)$ given by (2.17) is monotonically increasing in $\mathcal{E} \in (0, \mathcal{E}_\omega)$ such that*

$$T_{\text{even}}(0,\omega) = 2\pi\sqrt{\frac{1-\omega}{2\omega}}, \quad \lim_{\mathcal{E}\to\mathcal{E}_\omega} T_{\text{even}}(\mathcal{E},\omega) = +\infty,$$

Proof. Lemma 2.3 implies that $P(t) > 0$ for $t \in (0,1)\backslash\{\omega\}$, which yields $I''(\phi) > 0$ for $\phi \in (0,1)$ by (2.4). Since $V(\phi) \geq 0$ and $V(\sqrt{\omega}) = 0$, we can apply the main theorem from [5] by using the translated coordinate $\varphi = \phi - \sqrt{\omega}$. Since $I''(\phi) > 0$, the main theorem of [5] states that the period function $T_{\rm even}(\mathcal{E},\omega)$ is monotonically increasing in $\mathcal{E} \in (0,\mathcal{E}_\omega)$ for every $\omega \in (0,1)$. The limit for $T_{\rm even}(0,\omega)$ as $\mathcal{E} \to 0$ follows from the linearization of the center point $(\sqrt{\omega},0)$ of the system (2.1). The divergence of $T_{\rm even}(\mathcal{E},\omega)$ as $\mathcal{E} \to \mathcal{E}_\omega$ follows from the infinite period of the homoclinic orbit to the saddle equilibrium point $(0,0)$. $\square$

2.2 Monotonicity of Odd Periodic Waves

The period function $T(\mathcal{E},\omega)$ given by (1.11) can be rewritten for the odd periodic waves explicitly by

$$T_{\rm odd}(\mathcal{E},\omega) = 4\int_0^M \frac{d\phi}{\sqrt{2\mathcal{E} + (1-\omega)\log(1-\phi^2) - (1-\omega)\log(1-\omega) + \phi^2 - \omega}}, \tag{2.19}$$

where

$$M := -\min_{x\in[-\frac{L}{2},\frac{L}{2}]} \phi(x) \in (0,\sqrt{\omega}) = \max_{x\in[-\frac{L}{2},\frac{L}{2}]} \phi(x) \in (\sqrt{\omega},1) \tag{2.20}$$

is a positive root of $V(\phi) = \mathcal{E}$ for $\mathcal{E} \in (\mathcal{E}_\omega,\infty)$. The proof of monotonicity of the period function in $\mathcal{E}$ is easier for the odd periodic waves. The following proposition justifies the result stated in Theorem 1.3.

Proposition 2.5. *For every $\omega \in (0,1)$, the period function $T_{\rm odd}(\mathcal{E},\omega)$ given by (2.19) is monotonically decreasing in $\mathcal{E} \in (\mathcal{E}_\omega,\infty)$ such that*

$$\lim_{\mathcal{E}\to\mathcal{E}_\omega} T_{\rm odd}(\mathcal{E},\omega) = +\infty, \quad \lim_{\mathcal{E}\to\infty} T_{\rm odd}(\mathcal{E},\omega) = 0.$$

Proof. Using the same transformation $t = \phi^2$ as in Section 2.1, we rewrite $V(\phi)$ in (2.2) as

$$W(t) := \frac{1}{2}(\omega - t) + \frac{1}{2}(1-\omega)\log\frac{1-\omega}{1-t}, \quad t \in (0,1).$$

Similarly, we redefine $M \in (\sqrt{\omega},1)$ as $q := M^2 \in (\omega,1)$. Since $\mathcal{E} = V(M) = W(q)$, we use the change of variables $t = \phi^2$ for $t \in (0,q)$ and $t = qu$ for $u \in (0,1)$ and rewrite the integral (2.19) in the equivalent form:

$$\begin{aligned} T_{\rm odd}(\mathcal{E},\omega) &= 4\int_0^M \frac{d\phi}{\sqrt{2[V(M)-V(\phi)]}} \\ &= \sqrt{2}\int_0^q \frac{dt}{\sqrt{t[W(q)-W(t)]}} \\ &= \int_0^1 \frac{\sqrt{2q}}{\sqrt{u[W(q)-W(qu)]}}\,du. \end{aligned}$$

Since $V'(\phi) > 0$ for $\phi \in (\sqrt{\omega}, 1)$, we have $W'(q) > 0$. The chain rule

$$\frac{\partial T_{\rm odd}}{\partial \mathcal{E}} = \frac{\partial T_{\rm odd}}{\partial q} \left(\frac{\partial \mathcal{E}}{\partial q} \right)^{-1} = \frac{1}{W'(q)} \frac{\partial T_{\rm odd}}{\partial q}$$

implies that for a fixed $\omega \in (0,1)$, monotonicity of $T_{\rm odd}(\mathcal{E},\omega)$ in $\mathcal{E}$ and q coincide. Although the integral for $T_{\rm odd}(\mathcal{E},\omega)$ is weakly singular at $u = 0$ and $u = 1$, the derivative of $T_{\rm odd}(\mathcal{E},\omega)$ in q yields also weakly singular integrals and, hence, it can be computed by pointwise differentiation as in

$$\frac{\partial T_{\rm odd}}{\partial q} = \frac{1}{\sqrt{2q}} \int_0^1 \frac{du}{\sqrt{u[W(q) - W(qu)]}} - \frac{\sqrt{2q}}{2} \int_0^1 \frac{W'(q) - uW'(qu)}{\sqrt{u[W(q) - W(qu)]^3}}\, du,$$

where the second integral remains weakly singular at $u = 1$ since $W(q) - W(uq) = \mathcal{O}(1-u)$ and $W'(q) - uW'(qu) = \mathcal{O}(1-u)$ as $u \to 1$. The function $W(t)$ is strictly convex since

$$W'(t) = \frac{t - \omega}{2(1-t)} \qquad W''(t) = \frac{1-\omega}{2(1-t)^2} > 0.$$

If $F(t) := tW'(t) - W(t)$, then $F'(t) = tW''(t) > 0$ for $t \in (0,1)$, so that $F(q) > F(qu)$ for every $u \in (0,1)$. This implies for $u \in (0,1)$ that

$$qW'(q) - W(q) > quW'(qu) - W(qu), \quad \Rightarrow \quad W(q) - W(qu) < q[W'(q) - qW'(qu)].$$

Since $\sqrt{u[W(q) - W(qu)]} > 0$ for $u \in (0,1)$, it follows that

$$\frac{1}{\sqrt{q}} \frac{1}{\sqrt{u[W(q) - W(qu)]}} < \sqrt{q} \frac{W'(q) - uW'(qu)}{\sqrt{u}\,[W(q) - W(qu)]^{3/2}}, \quad u \in (0,1),$$

which proves that

$$\frac{\partial T_{\rm odd}}{\partial q} < 0, \quad q \in (\omega, 1),$$

This yields the desired monotonicity in $\mathcal{E}$ by the chain rule. The divergence of $T_{\rm odd}(\mathcal{E},\omega)$ as $\mathcal{E} \to \mathcal{E}_\omega$ follows from the infinite period of the homoclinic orbit to the saddle equilibrium point $(0,0)$ The zero limit of $T_{\rm odd}(\mathcal{E},\omega)$ as $\mathcal{E} \to \mathcal{E}_\omega$ follows from (2.19) by the dominated convergence theorem since $M \in (\sqrt{\omega}, 1)$ is finite. □

Remark 2.6. *The result of Proposition 2.5 is true for $\omega \in (-\infty, 0)$ with the only change*

$$\lim_{\mathcal{E} \to \mathcal{E}_\omega} T_{\rm odd} = \frac{2\pi}{\sqrt{|\omega|}},$$

which is computed from the linearization of the center point $(0,0)$ for $\omega \in (-\infty, 0)$. All other computations are identical to the proof of Proposition 2.5.

3 Spectral Analysis & Constraint Minimization

In this section, we obtain the sharp energetic stability criterion in Theorem 1.4 by two steps. As a first step, we analyze the Morse and nullity indices of the Schrödinger operators $\mathcal{L}_\pm : H^2_{\rm per} \subset L^2_{\rm per} \to L^2_{\rm per}$ given by (1.12), where the Morse index denoted by $n(\mathcal{L}_\pm)$ is the number of negative eigenvalues with the account of their multiplicities and the nullity index denoted by $z(\mathcal{L}_\pm)$ is the multiplicity of the zero eigenvalue. For the even wave, we prove that $n(\mathcal{L}_+) = z(\mathcal{L}_+) = z(\mathcal{L}_-) = 1$ and $n(\mathcal{L}_-) = 0$. For the odd wave, we prove that $n(\mathcal{L}_+) = 2$, $n(\mathcal{L}_-) = z(\mathcal{L}_+) = z(\mathcal{L}_-) = 1$. These results are discussed in Section 3.1.

As a second step, we analyze the Morse and nullity indices of the constrained operator $\mathcal{L}_+|_{\{\phi_0\}^\perp}$, where the constraint with $\phi_0 \equiv \frac{\phi}{1-\phi^2}$ is due to the fixed mass Q restriction [12]. We show that $n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 0$ and $z(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 1$ for the even wave if and only if the mapping $\omega \to Q(\phi)$ is monotonically inreasing, which yields Theorem 1.4 for the even wave. We also show that $n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 1$ and $z(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 1$ for the odd wave if and only if the mapping $\omega \to Q(\phi)$ is monotonically inreasing. This is still inconclusive for the energetic stability of the odd wave. However, restricting $H^1_{\rm per}$ to the space $\mathcal{Y}$ of odd perturbations with respect to the half-period results in $n(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = n(\mathcal{L}_-|_{\mathcal{Y}}) = 0$ and $z(\mathcal{L}_+|_{\{\phi_0\}^\perp \cap \mathcal{Y}}) = z(\mathcal{L}_-|_{\mathcal{Y}}) = 1$ if and only if the mapping $\omega \to Q(\phi)$ is monotonically increasing, which yields Theorem 1.4 for the odd wave. These results are described in Section 3.2. We note that the idea of restricting the space of periodic functions to odd perturbations with respect to the half-period is proposed in [11] for the stability analysis of odd waves in the cubic NLS equation.

3.1 Spectral Analysis near the Periodic Waves

Consider the Hessian operator $\mathcal{L} = H''(\phi) + \omega Q''(\phi)$ defined in (1.12) as an operator on $\mathbb{L}^2_{\rm per}$ with the domain in $\mathbb{H}^2_{\rm per}$. Since $\mathcal{L}$ is a diagonal composition of the Schrödinger operators $\mathcal{L}_+$ and $\mathcal{L}_-$ in $L^2_{\rm per}$ with the domain in $H^2_{\rm per}$, the spectrum of $\mathcal{L}$ is a superposition of the spectra of $\mathcal{L}_+$ and $\mathcal{L}_-$. Accordingly to [24], the spectrum of either $\mathcal{L}_+$ or $\mathcal{L}_-$ consists of an unbounded sequence of real eigenvalues

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 \;\; ... \;\; < \lambda_{2n-1} \leq \lambda_{2n} \;\; \cdots ,$$

where equality means that $\lambda_{2n-1} = \lambda_{2n}$ is a double eigenvalue. By [6, Theorem 3.1.2-(ii)], if φ is an eigenfunction associated to the eigenvalue λ_{2n-1} or λ_{2n}, then φ has exactly $2n$ zeroes on the periodic domain.

To characterize the Morse index denoted by $n(\mathcal{L}_\pm)$ and the nullity index denoted by $z(\mathcal{L}_\pm)$ of these operators $\mathcal{L}_+$ and $\mathcal{L}_-$, we use the following theorem which helps locate the zero eigenvalue in the spectrum, see [27, Theorem 3.1].

Proposition 3.1. *Let $\mathcal{M} = -\partial_x^2 + Q(x)$ be a general Schrödinger operator with the even, L-periodic, bounded potential Q and let $\{\varphi_1, \varphi_2\}$ be linearly independent solutions of $\mathcal{M}\varphi = 0$ satisfying*

$$\begin{cases} \varphi_1(0) = 1, \\ \varphi_1'(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \varphi_2(0) = 0, \\ \varphi_2'(0) = 1. \end{cases} \tag{3.1}$$

Assume that there exists $\theta \in \mathbb{R}$ such that

$$\varphi_1(x+L) = \varphi_1(x) + \theta\varphi_2(x), \quad \text{and} \quad \varphi_2(x+L) = \varphi_2(x), \tag{3.2}$$

and that the L-perodic eigenfunction φ_2 has two zeros on the periodic domain. The zero eigenvalue of $\mathcal{M}$ in L^2_{per} is simple if $\theta \neq 0$ and double if $\theta = 0$. It is the second eigenvalue of $\mathcal{M}$ if $\theta \geq 0$ and the third eigenvalue of $\mathcal{M}$ if $\theta < 0$.

3.1.1 Spectral analysis of even periodic waves

We show that $n(\mathcal{L}_+) = z(\mathcal{L}_+) = z(\mathcal{L}_-) = 1$ and $n(\mathcal{L}_-) = 0$. We proceed separately with the analysis of the Schrödinger operators $\mathcal{L}_+$ and $\mathcal{L}_-$ defined in (1.12) and computed at the even waves of Theorem 1.1 with the profile ϕ satisfying (1.9).

Proposition 3.2. *$n(\mathcal{L}_+) = z(\mathcal{L}_+) = 1$, that is, 0 is a simple eigenvalue of $\mathcal{L}_+$ associated with the eigenfunction ϕ', and there is only one negative eigenvalue, which is simple. In addition, the remainder of the spectrum of $\mathcal{L}_+$ in L^2_{per} consists of a discrete set of positive eigenvalues with finite multiplicities.*

Proof. On comparison with $\mathcal{M}$ in Proposition 3.1, we have

$$Q = 1 + (\omega - 1)\frac{1+\phi^2}{(1-\phi^2)^2}, \tag{3.3}$$

where $0 < \phi < 1$ is the spatial profile of the L-periodic orbit in Theorem 1.1 satisfying (1.9) with $x_0 = 0$ and $\omega \in (\omega_L, 1)$. Hence, Q is even, L-periodic, and bounded.

Consider the family of periodic orbits of the second-order equation (1.6) associated with the period function $T(\mathcal{E}, \omega)$ for the energy level $\mathcal{E} = E(\phi, \phi')$ of the first-order invariant (1.7) with $\mathcal{E} \in (0, \mathcal{E}_\omega)$. Due to monotonicity of the mapping $\mathcal{E} \to T(\mathcal{E}, \omega)$ for fixed $\omega \in (0, 1)$ in Theorem 1.3, there exists a unique $\mathcal{E} = \mathcal{E}_L(\omega)$ of $T(\mathcal{E}_L(\omega), \omega) = L$ for a fixed spatial period $L > 0$ and $\omega \in (\omega_L, 1)$. We further define $\phi_L(\omega) \in (0, 1)$ as a root of $V(\phi) = \mathcal{E}$ for $\mathcal{E} = \mathcal{E}_L(\omega)$. Two roots exist for the maximum and minimum of the spatial profile ϕ. Since

$$V'(\phi) = -\frac{\phi(\omega - \phi^2)}{1-\phi^2},$$

we have $V'(\phi_L(\omega)) \neq 0$ for either choice for $\phi_L(\omega)$. Equations (1.6) and (1.7) imply that

$$\phi''(0) = -V'(\phi_L(\omega)) \quad \text{and} \quad \left.\frac{\partial\phi(0)}{\partial\mathcal{E}}\right|_{\mathcal{E}=\mathcal{E}_L(\omega)} = \frac{1}{V'(\phi_L(\omega))}, \tag{3.4}$$

where the family of periodic orbits parametrized by $\mathcal{E}$ is restricted to even functions by using the translational invariance of the second-order equation (1.6).

Since $\mathcal{L}_+$ is a linearized operator for (1.6), we obtain two linearly independent solutions of $\mathcal{L}_+\varphi = 0$ in Proposition 3.1 by using

$$\varphi_1(x) = \left.\frac{\partial\phi(x)}{\partial\mathcal{E}}\right|_{\mathcal{E}=\mathcal{E}_L(\omega)} V'(\phi_L(\omega)), \quad \varphi_2(x) = -\frac{\phi'(x)}{V'(\phi_L(\omega))}. \tag{3.5}$$

Since ϕ is even, we obtain (3.1) from (3.4). The second solution φ_2 is L-periodic and has two zeros on periodic domain according to the assumption of Proposition 3.1. Computing the first solution φ_1 after the period L, we obtain

$$\varphi_1(L) = \left.\frac{\partial\phi(L)}{\partial\mathcal{E}}\right|_{\mathcal{E}=\mathcal{E}_L(\omega)} V'(\phi_L(\omega)) \quad \text{and} \quad \varphi_1'(L) = \left.\frac{\partial\phi'(L)}{\partial\mathcal{E}}\right|_{\mathcal{E}=\mathcal{E}_L(\omega)} V'(\phi_L(\omega)) =: \theta.$$

Since $\phi(T(\mathcal{E},\omega)) = \phi(0)$ and $\phi'(T(\mathcal{E},\omega)) = 0$, taking derivative of these equations in $\mathcal{E}$ at the energy level $\mathcal{E} = \mathcal{E}_L(\omega)$ implies that $\varphi_1(L) = 1$ and

$$\theta = -\left.\frac{\partial T}{\partial\mathcal{E}}\right|_{\mathcal{E}=\mathcal{E}_L(\omega)} \phi''(0)V'(\phi_L(\omega)) = \left.\frac{\partial T}{\partial\mathcal{E}}\right|_{\mathcal{E}=\mathcal{E}_L(\omega)} \left[V'(\phi_L(\omega))\right]^2,$$

where we have used (3.4) again. The L-periodicity of Q implies that φ_1 satisfies (3.2) with the sign of θ given by the sign of the derivative of the mapping $\mathcal{E} \to T(\mathcal{E},\omega)$ at $\mathcal{E} = \mathcal{E}_L(\omega)$. Since $\theta > 0$ by Proposition 2.4, Proposition 3.1 proves the assertion. $\square$

Remark 3.3. *Let $L > 0$ be fixed. Using the implicit function theorem and the fact that* $\mathrm{Ker}(\mathcal{L}_+) = \mathrm{Span}(\phi')$ *with ϕ' being odd, it is possible to prove that the mapping $\omega \mapsto \phi \in H^2_{\rm per,e}$ is C^1 for every $\omega \in (\omega_L, 1)$. In addition, differentiating (1.6) with respect to ω yields the derivative equation:*

$$\mathcal{L}_+\left(\frac{d\phi}{d\omega}\right) = -\frac{\phi}{1-\phi^2}. \tag{3.6}$$

Note that the mapping $\omega \mapsto \phi \in H^2_{\rm per,e}$ is only stated to be continuous for every $\omega \in (\omega_L, 1)$ in [34, Proposition 2.6], so (3.6) is an improvement.

For the odd wave case, the use of Proposition 3.1 is not necessary, and we can use the standard Floquet theory in [24] and the argument is simpler.

Proposition 3.4. *$n(\mathcal{L}_-) = 0$ and $z(\mathcal{L}_-) = 1$, that is, 0 is a simple eigenvalue of $\mathcal{L}_-$ associated with the eigenfunction ϕ and the remainder of the spectrum of $\mathcal{L}_-$ in $L^2_{\rm per}$ consists of a discrete set of positive eigenvalues with finite multiplicities.*

Proof. Since $0 < \phi < 1$ we obtain from the definition of $\mathcal{L}_-$ that

$$\mathcal{L}_- = -\partial_x^2 + \frac{\omega - \phi^2}{1-\phi^2}.$$

Since ϕ is positive and satisfies (1.5), we obtain that $\mathcal{L}_-\phi = 0$. By standard Floquet theory in [24], we deduce that zero is the first eigenvalue of $\mathcal{L}_-$ which is simple. Again, the last part of the proposition is obtained from the fact that $\mathcal{L}_-$ is a self-adjoint operator and the compact embedding $H^2_{per} \hookrightarrow L^2_{per}$. $\square$

By Proposition 3.2 and 3.4, we summarize the straightforward result on zero eigenvalue and its eigenfunctions.

Corollary 3.5. *The Hessian operator $\mathcal{L}$ defined by (1.12) in $\mathbb{L}^2_{\rm per}$ with domain $\mathbb{H}^2_{\rm per}$ has one negative eigenvalue which is simple. Zero is a double eigenvalue with associated eigenfunctions $(\phi', 0)$ and $(0, \phi)$. In addition, the remainder of the spectrum consists of a discrete set of positive eigenvalues with finite multiplicities.*

3.1.2 Spectral analysis of odd periodic waves

We show that $n(\mathcal{L}_+) = 2$, $n(\mathcal{L}_-) = z(\mathcal{L}_+) = z(\mathcal{L}_-) = 1$. We proceed separately with the analysis of the Schrödinger operators $\mathcal{L}_+$ and $\mathcal{L}_-$ defined in (1.12) and computed at the odd waves of Theorem 1.1 with the profile ϕ satisfying (1.10).

Proposition 3.6. *$n(\mathcal{L}_+) = 2$ and $z(\mathcal{L}_+) = 1$, that is, 0 is a simple eigenvalue of $\mathcal{L}_+$ associated with the eigenfunction ϕ', and there are two negative simple eigenvalues. The remainder of the spectrum of $\mathcal{L}_+$ in L^2_{per} consists of a discrete set of positive eigenvalues with finite multiplicities.*

Proof. We can prove the assertion in two different ways. Here, we show the proof using Theorem 3.1 as a comparison to the proof of even waves. The other proof is presented in [34, Proposition 4.7]. The potential Q in the linear operator $\mathcal{M}$ of Proposition 3.1 is defined by the same expression (3.3), where $-1 < \phi < 1$ is the spatial profile of the L-periodic orbit in Theorem 1.1 satisfying (1.10) with $x_0 = 0$ and $\omega \in (\Omega_L, 1)$. Hence, Q is even, L-periodic, and bounded. Since ϕ is even with respect to $x = \frac{L}{4}$ due to the second property in (1.10), Q has the minimum period $\frac{L}{2}$ and it is also even with respect to $x = \frac{L}{4}$. Therefore, we can repeat the proof of Proposition 3.2 and introduce the family of odd periodic orbits for the energy level $\mathcal{E} = E(\phi, \phi')$ with $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$. Again, due to monotonicity of the mapping $\mathcal{E} \to T(\mathcal{E}, \omega)$ for fixed $\omega \in (-\infty, 1)$ in Theorem 1.3, there exists a unique $\mathcal{E} = \mathcal{E}_L(\omega)$ of $T(\mathcal{E}_L(\omega), \omega) = L$ for a fixed spatial period $L > 0$ and $\omega \in (\Omega_L, 1)$. We further define $\phi_L(\omega) \in (0, 1)$ as a unique root of $V(\phi) = \mathcal{E}$ for $\mathcal{E} = \mathcal{E}_L(\omega)$, with the same property (3.4) and the same definition (3.5) of two solutions of $\mathcal{L}_+\varphi = 0$.

To satisfy the initial data in (3.1) for the two solutions, we can use the translational invariance of the second-order equation (1.6) and translate the family of odd periodic orbits to the family of even periodic orbits by

$$\phi(x) \to \phi\left(x - \frac{1}{4}T(\mathcal{E}, \omega)\right). \tag{3.7}$$

Then, assumptions of Proposition 3.1 are satisfied and the second solution φ_2 is L-periodic and has two zeros on periodic domain, whereras the first solution φ_1 satisfies (3.2) with the same definition of θ:

$$\theta = \frac{\partial T}{\partial \mathcal{E}}\bigg|_{\mathcal{E}=\mathcal{E}_L(\omega)} \left[V'(\phi_L(\omega))\right]^2.$$

Since $\theta < 0$ by Proposition 2.5, Proposition 3.1 proves the assertion for every $\omega \in (\Omega_L, 1)$. $\square$

Remark 3.7. *Let $L > 0$ be fixed. Using the implicit function theorem and the fact that $\text{Ker}(\mathcal{L}_{+,\text{e}}) = \text{Span}(\phi')$ it is possible to prove again that the mapping $\omega \mapsto \phi \in H^2_{\text{per,o}}$ is C^1 for every $\omega \in (\Omega_L, 1)$ with the same derivative equation (3.6).*

Proposition 3.8. *$n(\mathcal{L}_-) = 1$ and $z(\mathcal{L}_-) = 1$, that is, 0 is a simple eigenvalue of $\mathcal{L}_-$ associated with the eigenfunction ϕ, and there is only one negative eigenvalue, which is simple. The remainder of the spectrum of $\mathcal{L}_-$ in L^2_{per} consists of a discrete set of positive eigenvalues with finite multiplicities.*

Proof. On comparison with $\mathcal{M}$ in Proposition 3.1, we have

$$Q = 1 + \frac{\omega - 1}{1 - \phi^2}, \tag{3.8}$$

where $-1 < \phi < 1$ for every $\omega \in (\Omega_L, 1)$. Similarly to proof of Proposition 3.6, the L-peiodic and bounded Q in (3.8) is even with respect to both $x = 0$ and $x = \frac{L}{4}$ and has the minimal period $\frac{L}{2}$. After the translation (3.7) with $\mathcal{E} = \mathcal{E}_L(\omega)$, the lowest eigenvalue of $\mathcal{L}_+$ in $L^2_{\text{per,o}}$ is at zero, associated with the translated eigenfunction

$$\phi'(x) \to \phi'\left(x - \frac{L}{4}\right), \tag{3.9}$$

which is now odd. It follows from the relation between $\mathcal{L}_-$ and $\mathcal{L}_+$:

$$\mathcal{L}_- = \mathcal{L}_+ + \frac{2(1-\omega)\phi^2}{(1-\phi^2)^2}, \qquad \omega < 1 \tag{3.10}$$

that the lowest eigenvalue of $\mathcal{L}_-$ in $L^2_{\text{per,o}}$ is greater than the lowest eigenvalue of $\mathcal{L}_+$ in $L^2_{\text{per,o}}$. Therefore, $\mathcal{L}_-$ is strictly positive in $L^2_{\text{per,o}}$.

To study eigenvalues of $\mathcal{L}_-$ in $L^2_{\text{per,o}}$ after the translation (3.7) with $\mathcal{E} = \mathcal{E}_L(\omega)$, we note that $\mathcal{L}_-$ has the zero eigenvalue in $L^2_{\text{per,e}}$ associated with the translated eigenfunction

$$\phi(x) \to \phi\left(x - \frac{L}{4}\right),$$

which is now even. Since this eigenfunction for the zero eigenvalue of $\mathcal{L}_-$ in $L^2_{\text{per,e}}$ has two zeros on the periodic domain, there exists a negative eigenvalue of $\mathcal{L}_-$ in $L^2_{\text{per,e}}$ and by Theorem 3.1, zero is the second simple eigenvalue of $\mathcal{L}_-$ in $L^2_{\text{per,e}}$. Combining with positivity $\mathcal{L}_-$ in $L^2_{\text{per,e}}$, we have the assertion. □

By Propositions 3.6 and 3.8, we summarize the straightforward result on zero eigenvalue its eigenfunctions.

Corollary 3.9. *The Hessian operator $\mathcal{L}$ defined by (1.12) in $\mathbb{L}^2_{\text{per}}$ with domain $\mathbb{H}^2_{\text{per}}$ has three negative eigenvalue, which are semi-simple. Zero is a double eigenvalue with associated eigenfunctions $(\phi', 0)$ and $(0, \phi)$. In addition, the remainder of the spectrum consists of a discrete set of positive eigenvalues with finite multiplicities.*

3.2 Constraint Energy Minimization of Periodic Waves

For the wave profile $\phi \in H^1_{\text{per}}$ given by either even or odd periodic wave in Theorem 1.1, we can define the energy $H(\phi)$ and mass $Q(\phi)$ computed from (1.3) and (1.4). We recall from Remarks 3.3 and 3.7 that the mapping $\omega \to \phi \in H^1_{\text{per}}$ is C^1 for either even or odd periodic wave. Since $\phi \in H^1_{\rm per}$ is a critical point of the augmented energy functional $G(u)$ given by (1.8), we have

$$\frac{d}{d\omega}G(\phi) = \frac{d}{d\omega}H(\phi) + \omega\frac{d}{d\omega}Q(\phi) + Q(\phi) = Q(\phi),$$

which implies that the mapping $\omega \to G(\phi)$ is C^2 and

$$\frac{d^2}{d\omega^2}G(\phi) = \frac{d}{d\omega}Q(\phi) = 2\langle \frac{\phi}{1-\phi^2}, \frac{d\phi}{d\omega}\rangle.$$

By Corollaries 3.5 and 3.9, the Morse index for the Hessian operator $\mathcal{L} = H''(\phi) + \omega Q''(\phi)$ in (1.12) is nonzero so that $\phi \in H^1_{\rm per}$ is a saddle point of $G(u)$. We further clarify if $\phi \in H^1_{\rm per}$ is a local minimizer of energy $H(u)$ under the constraint of fixed mass $Q(u)$, which is degenerate only due to symmetries.

The NLS–IDD equation (1.1) can be formulated as a Hamiltonian system in the coordinate $u = p + iq$ with $(p,q) \in \mathbb{H}^1_{\rm per}$. The two basic symmetries of the NLS–IDD equation (1.1) are the translation and rotation symmetries. If $u = u(t,x)$ is a solution, so are $e^{-i\theta}u(t,x)$ and $u(t, x-\xi)$ for any $\theta, \xi \in \mathbb{R}$. Considering $u = p + iq$, this yields the invariance under the two transformations given by

$$S_1(\theta)\begin{bmatrix} p \\ q \end{bmatrix} := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} \tag{3.11}$$

and

$$S_2(\xi)\begin{bmatrix} p \\ q \end{bmatrix} := \begin{bmatrix} p(\cdot - \xi, \cdot) \\ q(\cdot - \xi, \cdot) \end{bmatrix}. \tag{3.12}$$

A standing wave solution of the form $u(x,t) = e^{i\omega t}\phi(x)$ is given by

$$S(\omega t)\begin{bmatrix} \phi(x) \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix}\phi(x).$$

The actions S_1 and S_2 in (3.11) and (3.12) define unitary groups in $\mathbb{H}^1_{\rm per}$ with infinitesimal generators given by

$$S_1'(0) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad S_2'(0) = \partial_x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Separating the variables for the perturbation as

$$u(x,t) = e^{i\omega t}\left(\phi(x) + p(x,t) + iq(x,t)\right)$$

we obtain the two-dimensional kernel of the Hessian operator (1.12) spanned by the two symmetry transformations:

$$S_1'(0)\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ \phi \end{bmatrix} \quad \text{and} \quad S_2'(0)\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \phi' \\ 0 \end{bmatrix},$$

These symmetry modes agree with the eigenfunctions in $\mathrm{Ker}(\mathcal{L})$ given by Corollaries 3.5 and 3.9.

If we consider variation of energy $E(u)$ under fixed mass $Q(u)$, then we define the linear constraint on the real part of the perturbation:

$$\langle \phi_0, p \rangle_{L^2_{\rm per}} = 0, \quad \phi_0 \equiv \frac{\phi}{1-\phi^2}, \tag{3.13}$$

The Morse index of $\mathcal{L}_+$ changes under the constraint and we study how it changes separately for the even and odd periodic waves.

3.2.1 Constraint Energy Minimization of Even Periodic Solutions

Under the constraint (3.13), we define the Morse and nullity indices of the constrained operator $\mathcal{L}_+|_{\{\phi_0\}^\perp}$ and denote them by $n(\mathcal{L}_+|_{\{\phi_0\}^\perp})$ and $z(\mathcal{L}_+|_{\{\phi_0\}^\perp})$.

Proposition 3.10. *$n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 0$ and $z(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = 1$ if and only if the mapping $\omega \to Q(\phi)$ is monotonically increasing at $\omega \in (\omega_L, 1)$.*

Proof. Since $\langle \phi_0, \phi' \rangle_{L^2_{\rm per}} = 0$, we have $\phi' \in \mathrm{Ker}(\mathcal{L}_+|_{\{\phi_0\}^\perp})$ by Proposition 3.2. It follows by [12, Theorem 2.7] that

$$n(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = n(\mathcal{L}_+) - 1 = 0, \quad z(\mathcal{L}_+|_{\{\phi_0\}^\perp}) = z(\mathcal{L}_+) = 1$$

if and only if

$$\langle \mathcal{L}_+^{-1} \phi_0, \phi_0 \rangle_{L^2_{\rm per}} < 0,$$

where equation (3.6) implies that

$$\langle \mathcal{L}_+^{-1} \phi_0, \phi_0 \rangle_{L^2_{\rm per}} = -\left\langle \phi_0, \frac{d\phi}{d\omega} \right\rangle_{L^2_{\rm per}} = -\frac{1}{2}\frac{d}{d\omega} Q(\phi) = -\frac{1}{2}\frac{d^2}{d\omega^2} G(\phi).$$

This completes the proof of the assertion. $\square$

Propositions 3.4 and (3.10) imply the following result, which yields the assertion of Theorem 1.4 for even periodic waves.

Corollary 3.11. *The Hessian operator $\mathcal{L}$ defined by (1.12) in $\mathbb{L}^2_{\rm per}$ with domain $\mathbb{H}^2_{\rm per}$ under the constraint (3.13) is non-negative and admits a double zero eigenvalue with associated eigenfunctions $(\phi', 0)$ and $(0, \phi)$ if and only if the mapping $\omega \to Q(\phi)$ is monotonically increasing at $\omega \in (\omega_L, 1)$.*

3.2.2 Constraint Energy Minimization of Odd Periodic Solutions

For the odd waves, the constraint (3.13) is not sufficient to remove all three negative directions. We recall the definition (1.13) for $\mathcal{Y} \subset H^1_{\rm per}$ spanned by functions which are odd with respect to the half-period. We define the Morse and nullity indices of the constrained operator $\mathcal{L}_-$ and denote them by $n(\mathcal{L}_-|_{\mathcal{Y}})$ and $z(\mathcal{L}_-|_{\mathcal{Y}})$. Under the additional constraint (3.13), we define the Morse and nullity indices of the constrained operator $\mathcal{L}_+|_{\{\phi_0\}^\perp\cap\mathcal{Y}}$ and denote them by $n(\mathcal{L}_+|_{\{\phi_0\}^\perp\cap\mathcal{Y}})$ and $z(\mathcal{L}_+|_{\{\phi_0\}^\perp\cap\mathcal{Y}})$.

Proposition 3.12. *$n(\mathcal{L}_+|_{\{\phi_0\}^\perp\cap\mathcal{Y}}) = z(\mathcal{L}_+|_{\{\phi_0\}^\perp\cap\mathcal{Y}}) = 0$ if and only if the mapping $\omega \to Q(\phi)$ is monotonically increasing at $\omega \in (\Omega_L, 1)$. Furthermore, it has $n(\mathcal{L}_-|_{\mathcal{Y}}) = 0$ and $z(\mathcal{L}_-|_{\mathcal{Y}}) = 1$.*

Proof. Since $\phi' \notin \mathcal{Y}$ and $\phi \in \mathcal{Y}$, we have $\phi' \notin \mathrm{Ker}(\mathcal{L}_+|_{\mathcal{Y}})$ and $\phi \in \mathrm{Ker}(\mathcal{L}_-|_{\mathcal{Y}})$ so that $z(\mathcal{L}_+|_{\mathcal{Y}}) = 0$ and $z(\mathcal{L}_-|_{\mathcal{Y}}) = 1$. Since the eigenfunctions of $\mathcal{L}_+$ and $\mathcal{L}_-$ for the smallest (negative) eigenvalue are even with respect to the half-period, we also have $n(\mathcal{L}_+|_{\mathcal{Y}}) = 1$ and $n(\mathcal{L}_-|_{\mathcal{Y}}) = 0$. In addition, we have $\phi_0 \in \mathcal{Y}$. It follows by [12, Theorem 2.7] that

$$n(\mathcal{L}_+|_{\{\phi_0\cap\mathcal{Y}\}^\perp}) = n(\mathcal{L}_+|_{\mathcal{Y}}) - 1 = 0, \quad z(\mathcal{L}_+|_{\{\phi_0\cap\mathcal{Y}\}^\perp}) = z(\mathcal{L}_+|_{\mathcal{Y}}) = 0$$

if and only if

$$\langle \mathcal{L}_+^{-1}\phi_0, \phi_0\rangle_{L^2_{\rm per}} < 0,$$

where equation (3.6) implies again that

$$\langle \mathcal{L}_+^{-1}\phi_0, \phi_0\rangle_{L^2_{\rm per}} = -\left\langle \phi_0, \frac{d\phi}{d\omega}\right\rangle_{L^2_{\rm per}} = -\frac{1}{2}\frac{d}{d\omega}Q(\phi) = -\frac{1}{2}\frac{d^2}{d\omega^2}G(\phi).$$

This completes the proof of the assertion. □

Proposition (3.12) implies the following result, which yields the assertion of Theorem 1.4 for odd periodic waves.

Corollary 3.13. *The Hessian operator $\mathcal{L}$ defined by (1.12) in $\mathbb{L}^2_{\rm per}$ with domain $\mathbb{H}^2_{\rm per} \cap \mathcal{Y}$ under the constraint (3.13) is non-negative and admits a simple zero eigenvalue with the associated eigenfunction $(0, \phi)$ if and only if the mapping $\omega \to Q(\phi)$ is monotonically increasing at $\omega \in (\Omega_L, 1)$.*

4 Endpoint Monotonicity of Mass

In this section, we prove the mapping $\omega \to Q(\phi)$ is monotonically increasing near the left endpoints: $\omega = \omega_L$ for even waves and $\omega = \Omega_L$ for odd waves; and is monotonically decreasing near the right end point ($\omega = 1$ for both even and odd waves). The even wave case is discussed in Section 4.1 and the odd wave case is discussed in Section 4.2.

The left endpoint cases are straightforward corollaries of [34, Proposition 2.4, 2.8], and these two results are shown in Proposition 4.1 and 4.4. The right endpoint cases require delicate construction and analysis, we first prove two technical Lemmas 4.2 and 4.5 to derive formulas, and then the actual computational proofs are in Proposition 4.3 and 4.6.

4.1 Endpoint monotonicity of even waves

In this subsection, we will show that the mass functional is monotonically increasing near the left endpoint $\omega = \omega_L$ and monotonically decreasing near the right endpoint $\omega = 1$. The period and mass of even waves can be written as

$$T = 2\int_m^M \frac{1}{\sqrt{2(\mathcal{E}_L - V)}}\, d\phi, \qquad Q = -2\int_m^M \frac{\log(1-\phi^2)}{\sqrt{2(\mathcal{E}_L - V)}}\, d\phi. \tag{4.1}$$

The turning points are determined by the roots $\{m, M\}$ in the denominator of (4.1), and they solve the equation

$$2(\mathcal{E}_L - V) = (2\mathcal{E}_L - 2\mathcal{E}_\omega + 1) - (1-\phi^2) + (1-\omega)\log(1-\phi^2) = 0, \tag{4.2}$$

where the turning points m and M are defined in (2.18). We first prove the left endpoint case.

Proposition 4.1. *For fixed $L > \frac{2\pi}{3^{1/4}}$, the even wave mapping $\omega \mapsto Q$ is monotonically increasing near $\omega = \omega_L^+$ where $\omega_L = \frac{2\pi^2}{L^2+2\pi^2}$.*

Proof. By [34, Proposition 2.4], there exists $a_0 > 0$, such that for every $a \in (-a_0, a_0)$, there is an even periodic solution $\phi \in H^2_{\text{per,e}}$. Let $s = \sqrt{\omega_L}$ and it can be decomposed as

$$\phi(x) = s + \psi(x), \quad s = \sqrt{\omega} - \frac{\omega_2}{2s}a^2 + \mathcal{O}(a^4) \quad \psi(x) = a\cos(kx) + a^2\left(\phi_2 + \frac{\omega_2}{2s}\right) + \mathcal{O}(a^3),$$

where expansion terms and constants are written as

$$\phi_2 = \frac{s^2+3}{12s(1-s^2)}(\cos(2kx) - 3), \quad \omega_2 = \frac{s^4+6s^2-9}{6(s^2-1)}, \quad k^2 = \frac{2s^2}{1-s^2}.$$

Let $Q_L(\omega) = Q(\phi(x,\omega))$ be the mass of the fixed period $L > 0$. Use the series expansion $-\log(1-z) = z + \frac{1}{2}z^2 + \mathcal{O}(z^3)$, let $z = \frac{2s\psi+\psi^2}{1-s^2}$ be the coordinate transformation, so the integrand of mass (1.4) can be written up to the third order, such that

$$-\log(1-\phi^2) = -\log(1-s^2) + \frac{2s}{1-s^2}\psi + \frac{1+s^2}{(1-s^2)^2}\psi^2 + \mathcal{O}(a^3),$$

so the mass can be computed as

$$\begin{aligned} Q_L(\omega) &= -L\log(1-s^2) + \frac{2s}{1-s^2}\int_{\mathbb{T}_L}\psi\,dx + \frac{1+s^2}{(1-s^2)^2}\int_{\mathbb{T}_L}\psi^2\,dx + \mathcal{O}(a^4) \\ &= -L\log(1-s^2) + \frac{3-6s^2-s^4}{6(1-s^2)^2}a^2L + \mathcal{O}(a^4) \\ &= -L\log(1-\omega_L) + \frac{L}{1-\omega_L}\left(\frac{3-6\omega_L-\omega_L^2}{9-6\omega_L-\omega_L^2}\right)(\omega-\omega_L) + \mathcal{O}\left((\omega-\omega_L)^2\right). \end{aligned}$$

Note that $Q_L'(\omega_L^+) > 0$ when $3-6\omega_L-\omega_L^2 > 0$ and this gives the critical length $\frac{2\pi}{3^{1/4}}$, as desired. ☐

For the right endpoint case, we start by constructing the following lemma to derive computation formula (4.3), and then the full proof is given in Proposition 4.3.

Lemma 4.2. *Fix $L > 0$ and let $\epsilon = 1-\omega$ and $\eta(\epsilon) = 1-m(\epsilon)^2$ be the parameter determined by the fixed period even branch $T(\epsilon,\eta(\epsilon)) = L$. Then, the right derivative of mass at $\epsilon = 0$ can be computed as*

$$\frac{d}{d\epsilon}Q(\epsilon,\eta(\epsilon))\bigg|_{\epsilon=0^+} = -\int_0^1 \partial_\epsilon q(0,\eta_*,t)\,dt - \frac{dQ_*}{dL}\int_0^1 \partial_\epsilon \ell(0,\eta_*,t)\,dt. \tag{4.3}$$

where the right partial derivatives are computed using

$$\ell(\epsilon,\eta,t) := \frac{\eta}{\sqrt{1-\eta t}\sqrt{\eta(1-t)+\epsilon\log t}}, \quad q(\epsilon,\eta,t) := \frac{\eta\log(\eta t)}{\sqrt{1-\eta t}\sqrt{\eta(1-t)+\epsilon\log t}}, \tag{4.4}$$

and evaluated at the point $(0,\eta_*,t)$ with $\eta_* = \tanh^2(\frac{L}{2})$.

Proof. Let $L > 0$ be fixed and $\epsilon = 1-\omega \geq 0$ so that $\omega \to 1^-$ is equivalent to $\epsilon \to 0^+$. By change of variable $t = \frac{1-\phi^2}{1-m^2}$ and denoting $\eta := 1-m^2 \in (\epsilon, 1)$, use the root m to represent the constant $(2\mathcal{E}_L - 2\mathcal{E}_\omega + 1)$ in (4.2) and write the square root on denominator of (4.1) as

$$\Lambda(\epsilon,\eta,t) := 2(\mathcal{E}_L - V) = \eta(1-t) + \epsilon\log t, \qquad t \in (0,1], \tag{4.5}$$

where the turning points have mappings $m \mapsto 1$ and $M \mapsto \frac{1-M^2}{1-m^2}$. Denote $\tau_\epsilon := \frac{1-M^2}{1-m^2}$ as the unique root of (4.5) on $t \in (0,1)$, so the period and mass in (4.1) can be written as

$$T(\epsilon,\eta) = \int_{\tau_\epsilon}^1 \ell(\epsilon,\eta,t)\,dt, \qquad Q(\epsilon,\eta) = -\int_{\tau_\epsilon}^1 q(\epsilon,\eta,t)\,dt, \qquad \epsilon > 0, \tag{4.6}$$

where the integrands are exactly (4.4) and they are represented as

$$\ell(\epsilon,\eta,t) = \frac{\eta}{\sqrt{1-\eta t}\sqrt{\Lambda(\epsilon,\eta,t)}}, \qquad q(\epsilon,\eta,t) = \frac{\eta\log(\eta t)}{\sqrt{1-\eta t}\sqrt{\Lambda(\epsilon,\eta,t)}}. \tag{4.7}$$

Note that functions (4.7) are integrable on $(\tau_\epsilon, 1)$ since $\ell, q \sim \mathcal{O}\left((t-\tau_\epsilon)^{-1/2}\right)$ near $t = \tau_\epsilon$ and $\ell, q \sim \mathcal{O}\left((t-1)^{-1/2}\right)$ near $t = 1$. At $\epsilon = 0$, define $\tau_0 := 0$ and recalling (5.1) implies $\eta_* := \eta(0) = \tanh^2(\frac{L}{2})$, so they can be considered as the right limits

$$\lim_{\epsilon\to 0^+} \tau_\epsilon = \tau_0, \qquad \lim_{\epsilon\to 0^+} \eta = \eta_*.$$

Our goal is to compute right derivative using limit definition as $\epsilon \to 0^+$ and $\eta(\epsilon) \to \eta_*$. Despite the ϵ-dependence, $\eta \in (0,1)$ is an independent variable in (4.7), so we define the compact set $I \subset (0,1)$, such that

$$I := \left[\frac{\eta_*}{2}, \frac{\eta_*+1}{2}\right] \subset (0,1), \qquad \eta_* = \tanh^2\left(\frac{L}{2}\right) \in I, \quad L > 0 \tag{4.8}$$

For sufficiently small $\epsilon > 0$, it suffices to construct uniform estimates for $\eta \in I$. Define the integrals (4.6) and integrands (4.7) with $\epsilon = 0$ and $\eta \in I$, such that

$$T(0,\eta) := \int_0^1 \ell(0,\eta,t)\,dt, \qquad Q(0,\eta) := -\int_0^1 q(0,\eta,t)\,dt, \qquad \epsilon = 0, \tag{4.9}$$

where

$$\ell(0,\eta,t) := \frac{\sqrt{\eta}}{\sqrt{1-\eta t}\sqrt{1-t}}, \qquad q(0,\eta,t) := \frac{\sqrt{\eta}\log(\eta t)}{\sqrt{1-\eta t}\sqrt{1-t}}. \tag{4.10}$$

Since the terms $(1-\eta t)^{-1/2}$ and $|\log(\eta t)|$ are bounded on $t \in (0,1)$, the integrands $\ell(0,\eta,t), q(0,\eta,t) \sim \mathcal{O}\left((1-t)^{-1/2}\right)$ near $t = 1$, and it implies that $\ell(0,\eta,t), q(0,\eta,t) \in L^1_t(0,1)$. For $\epsilon \geq 0$ and $\eta \in I$, use (4.6) and (4.9) to write the difference quotient of period and mass as

$$\frac{T(\epsilon,\eta) - T(0,\eta_*)}{\epsilon} = \int_{\tau_\epsilon}^1 \frac{\ell(\epsilon,\eta,t) - \ell(0,\eta_*,t)}{\epsilon}\,dt - \frac{1}{\epsilon}\int_0^{\tau_\epsilon} \ell(0,\eta_*,t)\,dt \tag{4.11}$$

$$\frac{Q(\epsilon,\eta) - Q(0,\eta_*)}{\epsilon} = -\int_{\tau_\epsilon}^1 \frac{q(\epsilon,\eta,t) - q(0,\eta_*,t)}{\epsilon}\,dt + \frac{1}{\epsilon}\int_0^{\tau_\epsilon} q(0,\eta_*,t)\,dt \tag{4.12}$$

and we would like to compute their limit as $\epsilon \to 0^+$.

First, we show that the second integrals in (4.11) and (4.12) vanishes, such that

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_0^{\tau_\epsilon} \ell(0,\eta_*,t)\,dt = 0, \qquad \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_0^{\tau_\epsilon} q(0,\eta_*,t)\,dt = 0, \tag{4.13}$$

and this relies on the fact that τ_ϵ is exponentially small. To see this, for $\epsilon > 0$, the function (4.5) has $\lim_{t\to 0^+} \Lambda = -\infty$ and $\Lambda(\epsilon,\eta,1) = 0$. Since $\partial_t \Lambda = 0$ at $t = \frac{\epsilon}{\eta}$ and $\partial_t^2 \Lambda = -\epsilon t^{-2} < 0$, the function (4.5) attains a unique maximum on $t \in (0,1)$. Thus, for every sufficiently small $\epsilon > 0$, the function (4.5) has $\Lambda(\epsilon,\eta,\frac{1}{2}) = \frac{1}{2}\eta - \epsilon\log 2 > 0$, which implies the root $\tau_\epsilon < \frac{1}{2}$, and it follows that

$$\tau_\epsilon = \exp\left[-\frac{\eta(1-\tau_\epsilon)}{\epsilon}\right] \leq e^{-\eta/(2\epsilon)}. \tag{4.14}$$

This can be used to construct bounds. Since $\tau_\epsilon < \frac{1}{2}$ and $\eta_* < 1$, the denominators of (4.10) has $\sqrt{(1-\eta_*\tau_\epsilon)(1-\tau_\epsilon)} > \frac{1}{2}$ and the period can be estimated as

$$\frac{1}{\epsilon}\int_0^{\tau_\epsilon} \ell(0,\eta_*,t)\,dt \leq \left(\frac{\tau_\epsilon}{\epsilon}\right)\frac{\sqrt{\eta_*}}{\sqrt{1-\eta_*\tau_\epsilon}\sqrt{1-\tau_\epsilon}} \leq \frac{2\tau_\epsilon}{\epsilon}. \tag{4.15}$$

Similarly, since $\sqrt{(1-\eta_*\tau_\epsilon)(1-\tau_\epsilon)} > \frac{1}{2}$ and $\sup_{\eta_*\in(0,1)}(2\sqrt{\eta_*}|\log\eta_*|) = \frac{4}{e} < 2$, the mass can be estimated as

$$\frac{1}{\epsilon}\int_0^{\tau_\epsilon} |q(0,\eta_*,t)|\,dt \leq \frac{1}{\epsilon}\int_0^{\tau_\epsilon} \left(2\sqrt{\eta_*}|\log\eta_*| + 2\sqrt{\eta_*}|\log t|\right)\,dt \leq \frac{2\tau_\epsilon}{\epsilon}\left(2+|\log\tau_\epsilon|\right). \tag{4.16}$$

As $\epsilon \to 0^+$, the inequality (4.14) implies the turning point $\tau_\epsilon \to 0^+$ exponentially fast, so the bounds in (4.15), (4.16) tends to zero and this verifies the limits in (4.13).

Thus, we only need to analyze the first integrals in (4.11) and (4.12). By the fundamental theorem of calculus, let $s \in [0,1]$ be a parameter so their integrands can be written as

$$\begin{aligned}\frac{\ell(\epsilon,\eta,t)-\ell(0,\eta_*,t)}{\epsilon} &= \frac{\ell(\epsilon,\eta,t)-\ell(0,\eta,t)}{\epsilon} + \frac{\ell(0,\eta,t)-\ell(0,\eta_*,t)}{\epsilon} \\ &= \int_0^1 \partial_\epsilon \ell(s\epsilon,\eta,t)\,ds + \frac{\eta-\eta_*}{\epsilon}\int_0^1 \partial_\eta \ell_0(\eta_*+s(\eta-\eta_*),t)\,ds,\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\frac{q(\epsilon,\eta,t)-q(0,\eta_*,t)}{\epsilon} &= \frac{q(\epsilon,\eta,t)-q(0,\eta,t)}{\epsilon} + \frac{q(0,\eta,t)-q(0,\eta_*,t)}{\epsilon} \\ &= \int_0^1 \partial_\epsilon q(s\epsilon,\eta,t)\,ds + \frac{\eta-\eta_*}{\epsilon}\int_0^1 \partial_\eta q_0(\eta_*+s(\eta-\eta_*),t)\,ds.\end{aligned} \tag{4.18}$$

The partial derivatives of three-variable functions $\ell(\epsilon,\eta,t)$ and $q(\epsilon,\eta,t)$ with respect to $\epsilon \in (0,1)$ can be computed as

$$\partial_\epsilon \ell(\epsilon,\eta,t) = -\frac{\eta\log t}{2\sqrt{1-\eta t}\,[\Lambda(\epsilon,\eta,t)]^{3/2}}, \quad \partial_\epsilon q(\epsilon,\eta,t) = -\frac{\eta\log(\eta t)\log t}{2\sqrt{1-\eta t}\,[\Lambda(\epsilon,\eta,t)]^{3/2}} \tag{4.19}$$

and they are evaluated at point $(s\epsilon,\eta,t)$ as the first integrands in (4.17) and (4.18). At $\epsilon = 0$, the partial derivatives of two-variable functions $\ell_0(\eta,t) := \ell(0,\eta,t)$ and $q_0(\eta,t) := q(0,\eta,t)$ with respect to $\eta \in (0,1)$ can be computed as

$$\partial_\eta \ell_0(\eta,t) = \frac{1}{2\sqrt{\eta(1-t)}\,(1-\eta t)^{3/2}}, \quad \partial_\eta q_0(\eta,t) = \frac{\log(\eta t)+2(1-\eta t)}{2\sqrt{\eta(1-t)}\,(1-\eta t)^{3/2}} \tag{4.20}$$

and they are evaluated at point $(\eta_*+s(\eta-\eta_*),t)$ to be the second integrands in (4.17) and (4.18).

The first quotients from (4.17) and (4.18) in integral representation read

$$\frac{\ell(\epsilon,\eta,t)-\ell(0,\eta,t)}{\epsilon} = \int_0^1 \partial_\epsilon \ell(s\epsilon,\eta,t)\,ds, \quad \frac{q(\epsilon,\eta,t)-q(0,\eta,t)}{\epsilon} = \int_0^1 \partial_\epsilon q(s\epsilon,\eta,t)\,ds, \tag{4.21}$$

which are then substituted into integrals (4.11) and (4.12). Let $\epsilon > 0$ be sufficiently small so it satisfies

$$\eta \in I, \qquad \tau_\epsilon < e^{-1/\sqrt{\epsilon}} < \frac{1}{2}\epsilon, \qquad \epsilon < \frac{\eta^2}{16}, \tag{4.22}$$

and we decompose the region of integration as

$$(\tau_\epsilon, 1) = (\tau_\epsilon, t_\epsilon) \cup \left(t_\epsilon, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right), \qquad t_\epsilon := e^{-\frac{1}{\sqrt{\epsilon}}}. \tag{4.23}$$

We prove that for any $\eta \in I$, the quotients (4.21) as integrands in (4.11) and (4.12) have the limits

$$\begin{aligned} \lim_{\epsilon \to 0^+} \int_{\tau_\epsilon}^1 \frac{\ell(\epsilon,\eta,t) - \ell(0,\eta,t)}{\epsilon}\, dt &= \int_0^1 \partial_\epsilon \ell(0,\eta,t)\, dt, \\ \lim_{\epsilon \to 0^+} \int_{\tau_\epsilon}^1 \frac{q(\epsilon,\eta,t) - q(0,\eta,t)}{\epsilon}\, dt &= \int_0^1 \partial_\epsilon q(0,\eta,t)\, dt. \end{aligned} \tag{4.24}$$

We separate $(\tau_\epsilon, 1)$ into three regions as (4.23) and analyze each region.

(i) <u>Region $(\tau_\epsilon, t_\epsilon)$:</u>
We show that the integrals vanish uniformly for $\eta \in I$, such that

$$\lim_{\epsilon \to 0^+} \int_{\tau_\epsilon}^{t_\epsilon} \frac{\ell(\epsilon,\eta,t) - \ell(0,\eta,t)}{\epsilon}\, dt = 0, \qquad \lim_{\epsilon \to 0^+} \int_{\tau_\epsilon}^{t_\epsilon} \frac{q(\epsilon,\eta,t) - q(0,\eta,t)}{\epsilon}\, dt = 0. \tag{4.25}$$

By triangle inequality, the integrals can be estimated as

$$\begin{aligned} \left| \int_{\tau_\epsilon}^{t_\epsilon} \frac{\ell(\epsilon,\eta,t) - \ell(0,\eta,t)}{\epsilon}\, dt \right| &\le \frac{1}{\epsilon} \int_{\tau_\epsilon}^{t_\epsilon} \left(|\ell(\epsilon,\eta,t)| + |\ell(0,\eta,t)| \right) dt, \\ \left| \int_{\tau_\epsilon}^{t_\epsilon} \frac{q(\epsilon,\eta,t) - q(0,\eta,t)}{\epsilon}\, dt \right| &\le \frac{1}{\epsilon} \int_{\tau_\epsilon}^{t_\epsilon} \left(|q(\epsilon,\eta,t)| + |q(0,\eta,t)| \right) dt, \end{aligned} \tag{4.26}$$

and we use (4.7) to construct the bounds.

For the term $|\ell(0,\eta,t)|$, since the mapping $t \mapsto (\sqrt{1-\eta t}\sqrt{1-t})^{-1}$ is monotonically increasing on $(0,1)$, the maximum over the interval $[\tau_\epsilon, t_\epsilon]$ is attained at $t = t_\epsilon$, and it follows that

$$\frac{1}{\epsilon} \int_{\tau_\epsilon}^{t_\epsilon} |\ell(0,\eta,t)|\, dt \le \frac{\sqrt{\eta}}{\sqrt{1-\eta t_\epsilon}\sqrt{1-t_\epsilon}} \left(\frac{t_\epsilon}{\epsilon}\right) =: C_{\ell,0} \left(\frac{t_\epsilon}{\epsilon}\right), \quad C_{\ell,0}(\epsilon,\eta) > 0. \tag{4.27}$$

Additionally, since $0 < \eta t \le \eta t_\epsilon < 1$, the mapping $t \mapsto |\log(\eta t)|$ is monotonically decreasing on $(0,1)$, so it has $\sup_{t \in [\tau_\epsilon, t_\epsilon]} |\log(\eta t)| = |\log(\eta \tau_\epsilon)|$ and the term $|q(0,\eta,t)|$ has the bound

$$\begin{aligned} \frac{1}{\epsilon} \int_{\tau_\epsilon}^{t_\epsilon} |q(0,\eta,t)|\, dt &\le \left(\frac{t_\epsilon - \tau_\epsilon}{\epsilon}\right) \frac{\sqrt{\eta}}{\sqrt{1-\eta t_\epsilon}\sqrt{1-t_\epsilon}} \left[|\log \eta| + \frac{\eta(1-\tau_\epsilon)}{\epsilon} \right] \\ &\le \frac{\sqrt{\eta}\left[\epsilon|\log\eta| + \eta(1-\tau_\epsilon)\right]}{\sqrt{1-\eta t_\epsilon}\sqrt{1-t_\epsilon}} \left(\frac{t_\epsilon}{\epsilon^2}\right) =: C_{q,0} \left(\frac{t_\epsilon}{\epsilon^2}\right), \end{aligned} \tag{4.28}$$

where $C_{q,0}(\epsilon,\eta) > 0$ is a constant.

For terms $|\ell(\epsilon,\eta,t)|$ and $|q(\epsilon,\eta,t)|$, the $\sqrt{\Lambda(\epsilon,\eta,t)}$ in the denominators of (4.7) stays positive on $t \in (\tau_\epsilon, t_\epsilon)$ and vanishes at $t = \tau_\epsilon$. Since $\eta < 1$ and small $\epsilon > 0$ is chosen to satisfy (4.22), it has $t < t_\epsilon < \frac{\epsilon}{2} \le \frac{\epsilon}{2\eta}$ which implies $-\eta \ge -\frac{\epsilon}{2t}$, so that $\partial_t \Lambda = -\eta + \frac{\epsilon}{t} \ge \frac{\epsilon}{2t}$. Use $\Lambda(\epsilon,\eta,\tau_\epsilon) = 0$ and the elementary inequality $\log x \ge \frac{x-1}{x}$ for $x = \frac{t}{\tau_\epsilon} \ge 1$, so the fundamental theorem of calculus gives

$$\Lambda(\epsilon,\eta,t) = \int_{\tau_\epsilon}^{t} \partial_r \Lambda(\epsilon,\eta,r)\, dr \ge \frac{\epsilon}{2} \int_{\tau_\epsilon}^{t} \frac{1}{r}\, dr = \frac{\epsilon}{2} \log\left(\frac{t}{\tau_\epsilon}\right) \ge \left(\frac{\epsilon}{2}\right) \frac{t - \tau_\epsilon}{t}$$

i.e. $\Lambda^{-1/2} \le \sqrt{\frac{2t}{\epsilon(t-\tau_\epsilon)}}$ for $t \in (\tau_\epsilon, t_\epsilon)$. Since the mapping $t \mapsto \frac{\sqrt{t}}{\sqrt{1-\eta t}}$ is increasing on $(0,1)$, the maximum over the interval $[\tau_\epsilon, t_\epsilon]$ is attained at $t = t_\epsilon$, and thus $|\ell(\epsilon,\eta,t)|$ has the bound

$$\begin{aligned} \frac{1}{\epsilon}\int_{\tau_\epsilon}^{t_\epsilon} |\ell(\epsilon,\eta,t)|\, dt &\le \frac{\eta}{\epsilon^{3/2}} \sqrt{\frac{2t_\epsilon}{1-\eta t_\epsilon}} \int_{\tau_\epsilon}^{t_\epsilon} \frac{1}{\sqrt{t-\tau_\epsilon}}\, dt \\ &= \frac{2\sqrt{2}\eta}{\sqrt{1-\eta t_\epsilon}} \sqrt{1 - \frac{\tau_\epsilon}{t_\epsilon}} \left(\frac{t_\epsilon}{\epsilon^{3/2}}\right) =: C_{\ell,1}\left(\frac{t_\epsilon}{\epsilon^{3/2}}\right), \quad C_{\ell,1}(\epsilon,\eta) > 0. \end{aligned} \tag{4.29}$$

By analogy to (4.28) and (4.29), with $|\log(\eta t)| \le |\log \eta| + |\log \tau_\epsilon|$, the term $|q(\epsilon,\eta,t)|$ can be bounded as

$$\frac{1}{\epsilon}\int_{\tau_\epsilon}^{t_\epsilon} |q(\epsilon,\eta,t)|\, dt \le \frac{2\sqrt{2}\eta\,[\epsilon|\log\eta| + \eta(1-\tau_\epsilon)]}{\sqrt{1-\eta t_\epsilon}} \sqrt{1 - \frac{\tau_\epsilon}{t_\epsilon}} \left(\frac{t_\epsilon}{\epsilon^{5/2}}\right) =: C_{q,1}\left(\frac{t_\epsilon}{\epsilon^{5/2}}\right). \tag{4.30}$$

where $C_{q,1}(\epsilon,\eta) > 0$ is some constant.

In conclusion, for some small $\epsilon_0 > 0$, we can take the supremum of coefficients in (4.27),(4.28),(4.29), and (4.30) to represent as a positive constant $C > 0$ that is uniformly bounded for $\epsilon \in (0,\epsilon_0)$ and $\eta \in I$, such that

$$C := \sup_{\substack{\epsilon\in(0,\epsilon_0)\\ \eta\in I}} \max\left\{C_{\ell,0}(\epsilon,\eta), C_{q,0}(\epsilon,\eta), C_{\ell,1}(\epsilon,\eta), C_{q,1}(\epsilon,\eta)\right\}.$$

where each terms in (4.26) can be further estimated as

$$\begin{aligned} \frac{1}{\epsilon}\int_{\tau_\epsilon}^{t_\epsilon} |\ell(0,\eta,t)|\, dt &\le C\left(\frac{t_\epsilon}{\epsilon}\right), & \frac{1}{\epsilon}\int_{\tau_\epsilon}^{t_\epsilon} |\ell(\epsilon,\eta,t)|\, dt &\le C\left(\frac{t_\epsilon}{\epsilon^{3/2}}\right), \\ \frac{1}{\epsilon}\int_{\tau_\epsilon}^{t_\epsilon} |q(0,\eta,t)|\, dt &\le C\left(\frac{t_\epsilon}{\epsilon^{2}}\right), & \frac{1}{\epsilon}\int_{\tau_\epsilon}^{t_\epsilon} |q(\epsilon,\eta,t)|\, dt &\le C\left(\frac{t_\epsilon}{\epsilon^{5/2}}\right). \end{aligned}$$

As $\epsilon \to 0^+$, since $\frac{t_\epsilon}{\epsilon^a} \to 0$ for any finite $a > 0$, the limits in (4.25) are justified.

(ii) Region $(t_\epsilon, \frac{1}{2})$:

We show that the quotients $\frac{\ell(\epsilon,\eta,t)-\ell(0,\eta,t)}{\epsilon}\chi_{[t_\epsilon,\frac{1}{2}]}(t)$, $\frac{q(\epsilon,\eta,t)-q(0,\eta,t)}{\epsilon}\chi_{[t_\epsilon,\frac{1}{2}]}(t) \in L^1_t(0,\frac{1}{2})$ for all $\eta \in I$, and they have the limits

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_{t_\epsilon}^{1/2} \frac{\ell(\epsilon,\eta,t)-\ell(0,\eta,t)}{\epsilon}\,dt &= \int_0^{1/2} \partial_\epsilon \ell(0,\eta,t)\,dt,\\ \lim_{\epsilon\to 0^+}\int_{t_\epsilon}^{1/2} \frac{q(\epsilon,\eta,t)-q(0,\eta,t)}{\epsilon}\,dt &= \int_0^{1/2} \partial_\epsilon q(0,\eta,t)\,dt.\end{aligned} \tag{4.31}$$

From (4.17) and (4.18), the quotients have the integral representation (4.21), so it suffices to show that $\chi_{\left[t_\epsilon,\frac{1}{2}\right]}(t)\partial_\epsilon \ell(s\epsilon,\eta,t), \chi_{\left[t_\epsilon,\frac{1}{2}\right]}(t)\partial_\epsilon q(s\epsilon,\eta,t) \in L^1_t(0,\frac{1}{2})$.

Since $t_\epsilon < t < \frac{1}{2}$ and $\eta < 1$, we have $1-t \geq \frac{1}{2}$ and $\log t < 0$ on $t \in (t_\epsilon, \frac{1}{2})$, so the function $\Lambda(s\epsilon,\eta,t), s\in[0,1]$ has the lower bound

$$\Lambda(s\epsilon,\eta,t) = \eta(1-t) + s\epsilon\log t \geq \frac{1}{2}\eta - \epsilon|\log t| \geq \frac{1}{2}\eta - \sqrt{\epsilon}.$$

By (4.22), the small $\epsilon > 0$ is chosen to satisfy $\sqrt{\epsilon} < \frac{\eta}{4}$, which gives a strictly positive bound $\Lambda(s\epsilon,\eta,t) \geq \frac{\eta}{4} > 0$. The partial derivatives are computed in (4.19), with $\sqrt{1-\eta t} \geq \frac{1}{\sqrt{2}}$ and $[\Lambda(s\epsilon,\eta,t)]^{-3/2} \leq \frac{8}{\eta^{3/2}}$, they are evaluated at the point $(s\epsilon,\eta,t)$ and estimated as

$$\begin{aligned}|\partial_\epsilon \ell(s\epsilon,\eta,t)| &= \frac{\eta|\log t|}{2\sqrt{1-\eta t}\,[\Lambda(s\epsilon,\eta,t)]^{3/2}} \leq \frac{4\sqrt{2}}{\sqrt{\eta}}|\log t|,\\ |\partial_\epsilon q(s\epsilon,\eta,t)| &= \frac{\eta|\log t||\log(\eta t)|}{2\sqrt{1-\eta t}\,[\Lambda(s\epsilon,\eta,t)]^{3/2}} \leq \frac{4\sqrt{2}}{\sqrt{\eta}}\left(|\log\eta||\log t| + |\log t|^2\right).\end{aligned}$$

Since $|\log t|, |\log t|^2 \in L^1_t(0,\frac{1}{2})$, we can define the majorant $g_1 \in L^1_t(0,\frac{1}{2})$, such that

$$g_1(t) = C_1\left(|\log t| + |\log t|^2\right), \quad C_1 := \sup_{\eta\in I}\left(\frac{4\sqrt{2}}{\sqrt{\eta}}\max\{1,|\log\eta|\}\right) < \infty$$

where $|\partial_\epsilon \ell(s\epsilon,\eta,t)| \leq g_1$ and $|\partial_\epsilon q(s\epsilon,\eta,t)| \leq g_1$ for all $t \in (0,\frac{1}{2})$, $s\in[0,1]$ and $\eta \in I$. This directly implies that $\partial_\epsilon \ell(s\epsilon,\eta,t), \partial_\epsilon q(s\epsilon,\eta,t) \in L^1_s(0,1)$ for each fixed $t \in (t_\epsilon,\frac{1}{2})$ since g_1 is independent of $s\in[0,1]$. As $\epsilon \to 0^+$, for each fixed $t\in(0,\frac{1}{2})$, $\eta \in I$ and $s\in[0,1]$, it has pointwise convergence $\Lambda(s\epsilon,\eta,t) \to \Lambda(0,\eta,t)$ a.e. so that $\partial_\epsilon \ell(s\epsilon,\eta,t) \to \partial_\epsilon \ell(0,\eta,t)$ and $\partial_\epsilon q(s\epsilon,\eta,t) \to \partial_\epsilon q(0,\eta,t)$ pointwise a.e. by (4.19). Since $\chi_{[t_\epsilon,\frac{1}{2}]}(t) \to 1$ pointwise a.e. as $\epsilon \to 0^+$ and $\partial_\epsilon \ell(s\epsilon,\eta,t), \partial_q \ell(s\epsilon,\eta,t) \in L^1_s(0,1)$, by dominated convergence theorem in $s\in(0,1)$, it has

$$\begin{aligned}\chi_{\left[t_\epsilon,\frac{1}{2}\right]}(t)\int_0^1 \partial_\epsilon \ell(s\epsilon,\eta,t)\,ds &\to \partial_\epsilon \ell(0,\eta,t),\\ \chi_{\left[t_\epsilon,\frac{1}{2}\right]}(t)\int_0^1 \partial_\epsilon q(s\epsilon,\eta,t)\,ds &\to \partial_\epsilon q(0,\eta,t),\end{aligned} \tag{4.32}$$

pointwise a.e. as $\epsilon \to 0^+$ and admits the majorant

$$\left|\chi_{[t_\epsilon,\frac{1}{2}]}(t)\int_0^1 \partial_\epsilon \ell(s\epsilon,\eta,t)\,ds\right| \le g_1, \quad \left|\chi_{[t_\epsilon,\frac{1}{2}]}(t)\int_0^1 \partial_\epsilon q(s\epsilon,\eta,t)\,ds\right| \le g_1.$$

Thus, use (4.21) to replace the integrals in (4.32) by the quotients and apply dominated convergence theorem again on $t \in (0,\frac{1}{2})$, so it follows that

$$\begin{aligned}
\lim_{\epsilon\to0^+}\int_{t_\epsilon}^{1/2} \frac{\ell(\epsilon,\eta,t)-\ell(0,\eta,t)}{\epsilon}\,dt &= \lim_{\epsilon\to0^+}\int_0^{1/2} \frac{\ell(\epsilon,\eta,t)-\ell(0,\eta,t)}{\epsilon}\chi_{[t_\epsilon,\frac{1}{2}]}(t)\,dt \\
&= \int_0^{1/2} \partial_\epsilon \ell(0,\eta,t)\,dt, \\
\lim_{\epsilon\to0^+}\int_{t_\epsilon}^{1/2} \frac{q(\epsilon,\eta,t)-q(0,\eta,t)}{\epsilon}\,dt &= \lim_{\epsilon\to0^+}\int_0^{1/2} \frac{q(\epsilon,\eta,t)-q(0,\eta,t)}{\epsilon}\chi_{[t_\epsilon,\frac{1}{2}]}(t)\,dt \\
&= \int_0^{1/2} \partial_\epsilon q(0,\eta,t)\,dt.
\end{aligned}$$

This proves the limits in (4.31).

(iii) <u>Region $(\frac{1}{2},1)$:</u>

We show that the quotient $\frac{\ell(\epsilon,\eta,t)-\ell(0,\eta,t)}{\epsilon}, \frac{q(\epsilon,\eta,t)-q(0,\eta,t)}{\epsilon} \in L^1(\frac{1}{2},1)$ and they satisfy the limits

$$\begin{aligned}
\lim_{\epsilon\to0^+}\int_{1/2}^{1} \frac{\ell(\epsilon,\eta,t)-\ell(0,\eta,t)}{\epsilon}\,dt &= \int_{1/2}^{1} \partial_\epsilon \ell(0,\eta,t)\,dt, \\
\lim_{\epsilon\to0^+}\int_{1/2}^{1} \frac{q(\epsilon,\eta,t)-q(0,\eta,t)}{\epsilon}\,dt &= \int_{1/2}^{1} \partial_\epsilon q(0,\eta,t)\,dt.
\end{aligned} \tag{4.33}$$

By the fundamental theorem of calculus, the quotients admits the integral representation (4.21), so it suffices to show that $\partial_\epsilon \ell(s\epsilon,\eta,t), \partial_\epsilon q(s\epsilon,\eta,t) \in L^1_t(\frac{1}{2},1)$ uniformly for $s \in [0,1]$ and $\eta \in I$.

Since $t \in (\frac{1}{2},1)$ and $0<\eta<1$, it has $1-\eta t>0$ and $|\log t| = \int_t^1 r^{-1}\,dr \le 2(1-t)$. For every $s\in[0,1]$ and sufficiently small $\epsilon>0$ chosen from (4.22), it has $2\epsilon < \frac{\eta^2}{8} < \frac{\eta}{2}$, which gives the strictly positive lower bound

$$\Lambda(s\epsilon,\eta,t) \ge \eta(1-t) - \epsilon|\log t| \ge (\eta-2\epsilon)(1-t) \ge \frac{\eta}{2}(1-t) > 0. \tag{4.34}$$

The partial derivatives are computed in (4.19), by using $1-\eta t > 1-\eta$, $|\log(\eta t)| \le |\log(\frac{\eta}{2})|$, $\frac{|\log t|}{1-t} \le 2$ and lower bound (4.34), they can be estimated as

$$\begin{aligned}
|\partial_\epsilon \ell(s\epsilon,\eta,t)| &= \frac{\eta|\log t|}{2\sqrt{1-\eta t}\,[\Lambda(s\epsilon,\eta,t)]^{3/2}} \le \frac{2\sqrt{2}}{\sqrt{\eta(1-\eta)}\sqrt{1-t}}, \\
|\partial_\epsilon q(s\epsilon,\eta,t)| &= \frac{\eta|\log t||\log(\eta t)|}{2\sqrt{1-\eta t}\,[\Lambda(s\epsilon,\eta,t)]^{3/2}} \le \frac{2\sqrt{2}}{\sqrt{\eta(1-\eta)}\sqrt{1-t}}\left|\log\left(\frac{\eta}{2}\right)\right|.
\end{aligned}$$

Since $(1-t)^{-1/2} \in L^1_t(\frac{1}{2},1)$, we can define the majorant $g_2 \in L^1_t(\frac{1}{2},1)$, such that

$$g_2(t) = \frac{C_2}{\sqrt{1-t}}, \qquad C_2 := \sup_{\eta \in I} \left(\frac{2\sqrt{2}}{\sqrt{\eta(1-\eta)}} \max\left\{1, \left|\log\left(\frac{\eta}{2}\right)\right|\right\} \right) < \infty$$

where $|\partial_\epsilon \ell(s\epsilon,\eta,t)| \le g_2$ and $|\partial_\epsilon q(s\epsilon,\eta,t)| \le g_2$ for all $t \in (\frac{1}{2},1)$, $s \in [0,1]$ and $\eta \in I$. This directly implies that $\partial_\epsilon \ell(s\epsilon,\eta,t), \partial_\epsilon q(s\epsilon,\eta,t) \in L^1_s(0,1)$ for each fixed $t \in (\frac{1}{2},1)$ since g_2 is independent of $s \in [0,1]$. As $\epsilon \to 0^+$, for each fixed $t \in (\frac{1}{2},1)$, $\eta \in I$ and $s \in [0,1]$, it has pointwise convergence $\Lambda(s\epsilon,\eta,t) \to \Lambda(0,\eta,t)$ a.e. so that $\partial_\epsilon \ell(s\epsilon,\eta,t) \to \partial_\epsilon \ell(0,\eta,t)$ and $\partial_\epsilon q(s\epsilon,\eta,t) \to \partial_\epsilon q(0,\eta,t)$ pointwise a.e. by (4.19). With $\partial_\epsilon \ell(s\epsilon,\eta,t), \partial_\epsilon q(s\epsilon,\eta,t) \in L^1_s(0,1)$, by dominated convergence theorem in $s \in (0,1)$, it follows that

$$\int_0^1 \partial_\epsilon \ell(s\epsilon,\eta,t)\,ds \to \partial_\epsilon \ell(0,\eta,t), \qquad \int_0^1 \partial_\epsilon q(s\epsilon,\eta,t)\,ds \to \partial_\epsilon q(0,\eta,t) \tag{4.35}$$

pointwise a.e. on $t \in (\frac{1}{2},1)$, and moreover

$$\left|\int_0^1 \partial_\epsilon \ell(s\epsilon,\eta,t)\,ds\right| \le g_2, \qquad \left|\int_0^1 \partial_\epsilon q(s\epsilon,\eta,t)\,ds\right| \le g_2.$$

Thus, use (4.21) to replace the integrals in (4.35) by the quotients and apply dominated convergence theorem again on $t \in (\frac{1}{2},1)$, so it proves the limits (4.33).

To summarize, the limits (4.25), (4.31) and (4.33) are uniform for $\eta \in I$ where I is chosen in (4.8). As $\epsilon \to 0^+$, since $\eta \to \eta_*$ and $\eta_* \in I$, it has $\partial_\epsilon \ell(0,\eta,t) \to \partial_\epsilon \ell(0,\eta_*,t)$ and $\partial_\epsilon q(0,\eta,t) \to \partial_\epsilon q(0,\eta_*,t)$ after passing the limit inside each integral by dominated convergence theorem, so that (4.24) becomes

$$\begin{aligned} \lim_{\epsilon\to 0^+} \int_{\tau_\epsilon}^1 \frac{\ell(\epsilon,\eta,t)-\ell(0,\eta,t)}{\epsilon}\,dt &= \int_0^1 \partial_\epsilon \ell(0,\eta_*,t)\,dt, \\ \lim_{\epsilon\to 0^+} \int_{\tau_\epsilon}^1 \frac{q(\epsilon,\eta,t)-q(0,\eta,t)}{\epsilon}\,dt &= \int_0^1 \partial_\epsilon q(0,\eta_*,t)\,dt. \end{aligned} \tag{4.36}$$

This finishes the analysis of the first quotients of (4.17) and (4.18) in (4.21).

It remains for us to consider the second quotients from (4.17) and (4.18) with integral representation

$$\begin{aligned} \frac{\ell(0,\eta,t)-\ell(0,\eta_*,t)}{\epsilon} &= \frac{\eta-\eta_*}{\epsilon}\int_0^1 \partial_\eta \ell_0(\zeta_s,t)\,ds, \\ \frac{q(0,\eta,t)-q(0,\eta_*,t)}{\epsilon} &= \frac{\eta-\eta_*}{\epsilon}\int_0^1 \partial_\eta q_0(\zeta_s,t)\,ds. \end{aligned} \tag{4.37}$$

where $\zeta_s := \eta_* + s(\eta-\eta_*) \in I$ and the partial derivatives in integrands are computed as (4.20) and evaluated at the point (ζ_s,t). We compute the limits $\epsilon \to 0^+$ of integrals in (4.37).

For every $t \in (0,1)$, $s \in [0,1]$, $\zeta_s \in I$ and sufficiently small $\epsilon > 0$ satisfying (4.22), it has $0 < 1-t < 1-\zeta_s t < 1$, which gives the bound

$$\begin{aligned} |\log(\zeta_s t)| + 2(1-\zeta_s) \le |\log t| + |\log \zeta_s| + 2 &\le (2 + |\log \zeta_s|)\left(1 + \frac{|\log t|}{2+|\log \zeta_s|}\right) \\ &\le (2+|\log\zeta_s|)(1+|\log t|). \end{aligned}$$

Thus, there exists positive constants

$$C_3 := \sup_{\zeta_s \in I} \frac{1}{2\sqrt{\zeta_s}(1-\zeta_s)^{3/2}} < \infty, \qquad C_4 := \sup_{\zeta_s \in I} \frac{|\log \zeta_s| + 2}{2\sqrt{\zeta_s}(1-\zeta_s)^{3/2}} < \infty.$$

which estimates the derivative as

$$\begin{aligned} |\partial_\eta \ell_0(\zeta_s, t)| &= \frac{1}{2\sqrt{\zeta_s(1-t)}(1-\zeta_s t)^{3/2}} \le \frac{C_3}{\sqrt{1-t}} =: g_3(t), \\ |\partial_\eta q_0(\zeta_s, t)| &\le \frac{|\log(\zeta_s t)| + 2(1-\zeta_s t)}{2\sqrt{\zeta_s(1-t)}(1-\zeta_s t)^{3/2}} \le \frac{C_4(1+|\log t|)}{\sqrt{1-t}} =: g_4(t). \end{aligned} \tag{4.38}$$

Note the majorant $g_3, g_4 \in L^1_s(0,1)$ and this shows that $\partial_\eta \ell_0(\zeta_s,t), \partial_\eta q_0(\zeta_s,t) \in L^1_s(0,1)$. For each fixed $t \in (0,1)$, as $\epsilon \to 0^+$, we have $\tau_\epsilon \to 0^+$ and $\zeta_s \to \eta_*$ as $\epsilon \to 0^+$. Since $\chi_{[\tau_\epsilon,1]}(t) \to 1$ pointwise a.e. on $t \in (0,1)$, by dominated convergence theorem on $s \in (0,1)$, it has

$$\chi_{[\tau_\epsilon,1]}(t) \int_0^1 \partial_\eta \ell_0(\zeta_s,t)\, ds \to \partial_\eta \ell_0(\eta_*,t), \quad \chi_{[\tau_\epsilon,1]}(t) \int_0^1 \partial_\eta q_0(\zeta_s,t)\, ds \to \partial_\eta q_0(\eta_*,t),$$

pointwise a.e. on $t \in (0,1)$, and they can be further majored as

$$\left| \chi_{[\tau_\epsilon,1]}(t) \int_0^1 \partial_\eta \ell_0(\zeta_s,t)\, ds \right| \le g_3(t), \quad \left| \chi_{[\tau_\epsilon,1]}(t) \int_0^1 \partial_\eta q_0(\zeta_s,t)\, ds \right| \le g_4(t).$$

Since $(1-t)^{-1/2}, (1+|\log t|)(1-t)^{-1/2} \in L^1_t(0,1)$, it follows that $g_3, g_4 \in L^1_t(0,1)$. By Lebesgue dominated convergence theorem on $t \in (0,1)$, rearrange (4.37) as the quotients in $\eta \in I$ and the limits can be computed as

$$\begin{aligned} \lim_{\eta \to \eta_*} \int_{\tau_\epsilon}^1 \frac{\ell(0,\eta,t) - \ell(0,\eta_*,t)}{\eta - \eta_*}\, dt &= \lim_{\epsilon \to 0^+} \int_{\tau_\epsilon}^1 \int_0^1 \partial_\eta \ell_0(\zeta_s,t)\, ds\, dt \\ &= \lim_{\epsilon \to 0^+} \int_0^1 \left[\chi_{[\tau_\epsilon,1]}(t) \int_0^1 \partial_\eta \ell_0(\zeta_s,t)\, ds \right] dt \\ &= \int_0^1 \partial_\eta \ell_0(\eta_*,t)\, dt, \\ \lim_{\eta \to \eta_*} \int_{\tau_\epsilon}^1 \frac{q(0,\eta,t) - q(0,\eta_*,t)}{\eta - \eta_*}\, dt &= \lim_{\epsilon \to 0^+} \int_{\tau_\epsilon}^1 \int_0^1 \partial_\eta q_0(\zeta_s,t)\, ds\, dt \\ &= \lim_{\epsilon \to 0^+} \int_0^1 \left[\chi_{[\tau_\epsilon,1]}(t) \int_0^1 \partial_\eta q_0(\zeta_s,t)\, ds \right] dt \\ &= \int_0^1 \partial_\eta q_0(\eta_*,t)\, dt. \end{aligned} \tag{4.39}$$

To finish, we synthesize all pieces. Since the period is fixed as $L > 0$, it has $T(\epsilon, \eta) = T(0, \eta_*) = L$. By the vanishing terms (4.13) and the limits (4.36) and (4.39), as $\epsilon \to 0^+$, take the limit of the period quotient (4.11) with integrand (4.17), and it follows that

$$\begin{aligned}
0 &= \lim_{\epsilon \to 0^+} \frac{T(\epsilon, \eta) - T(0, \eta_*)}{\epsilon} \\
&= \lim_{\epsilon \to 0^+} \int_{\tau_\epsilon}^1 \frac{\ell(\epsilon, \eta, t) - \ell(0, \eta_*, t)}{\epsilon}\, dt - \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^{\tau_\epsilon} \ell(0, \eta_*, t)\, dt \\
&= \int_0^1 \partial_\epsilon \ell(0, \eta_*, t)\, dt + \left(\lim_{\epsilon \to 0^+} \frac{\eta - \eta_*}{\epsilon} \right) \int_0^1 \partial_\eta \ell_0(\eta_*, t)\, dt
\end{aligned}$$

i.e.

$$\lim_{\epsilon \to 0^+} \frac{\eta - \eta_*}{\epsilon} = \frac{-\int_0^1 \partial_\epsilon \ell(0, \eta_*, t)\, dt}{\int_0^1 \partial_\eta \ell_0(\eta_*, t)\, dt}. \tag{4.40}$$

Similarly, by (4.13), (4.36) and (4.39), take $\epsilon \to 0^+$ of the mass quotient (4.12) and use (4.40), so the formal right derivative of $Q(\epsilon)$ at $\epsilon = 0$ can be computed as

$$\begin{aligned}
\left.\frac{dQ}{d\epsilon}\right|_{\epsilon = 0^+} &= \lim_{\epsilon \to 0^+} \frac{Q(\epsilon, \eta) - Q(0, \eta_*)}{\epsilon} \\
&= - \lim_{\epsilon \to 0^+} \int_{\tau_\epsilon}^1 \frac{q(\epsilon, \eta, t) - q(0, \eta_*, t)}{\epsilon}\, dt + \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^{\tau_\epsilon} q(0, \eta_*, t)\, dt \\
&= - \int_0^1 \partial_\epsilon q(0, \eta_*, t)\, dt - \left(\lim_{\epsilon \to 0^+} \frac{\eta - \eta_*}{\epsilon} \right) \int_0^1 \partial_\eta q_0(\eta_*, t)\, dt \\
&= - \int_0^1 \partial_\epsilon q(0, \eta_*, t)\, dt + \frac{\int_0^1 \partial_\eta q_0(\eta_*, t)\, dt}{\int_0^1 \partial_\eta \ell_0(\eta_*, t)\, dt} \int_0^1 \partial_\epsilon \ell(0, \eta_*, t)\, dt.
\end{aligned} \tag{4.41}$$

Next, we relate the integral quotient in (4.41) to derivative formula. Fix $\eta_* \in I$, for every $k \in \mathbb{R}$ such that $\eta = \eta_* + k \in I$, use (4.9) to write the difference quotients at $\epsilon = 0$ and apply fundamental theorem of calculus to a parameter $s \in [0, 1]$, such that

$$\begin{aligned}
\frac{T(0, \eta_* + k) - T(0, \eta_*)}{k} &= \int_0^1 \frac{\ell(0, \eta_* + k, t) - \ell(0, \eta_*, t)}{k}\, dt \\
&= \int_0^1 \int_0^1 \partial_\eta \ell_0(\eta_* + sk, t)\, ds\, dt, \\
\frac{Q(0, \eta_* + k) - Q(0, \eta_*)}{k} &= - \int_0^1 \frac{q(0, \eta_* + k, t) - q(0, \eta_*, t)}{k}\, dt \\
&= - \int_0^1 \int_0^1 \partial_\eta q_0(\eta_* + sk, t)\, ds\, dt.
\end{aligned} \tag{4.42}$$

Since the integrands are majorized by (4.38) and $g_3, g_4 \in L^1_s(0,1) \cap L^1_t(0,1)$, use limit (4.39) and apply dominated convergence theorem on $t \in (0,1)$, so the limits of (4.42) take the form

$$\begin{aligned} \lim_{k\to 0} \frac{T(0,\eta_*+k)-T(0,\eta_*)}{k} &= \int_0^1 \partial_\eta \ell_0(\eta_*,t)\,dt, \\ \lim_{k\to 0} \frac{Q(0,\eta_*+k)-Q(0,\eta_*)}{k} &= -\int_0^1 \partial_\eta q_0(\eta_*,t)\,dt. \end{aligned} \tag{4.43}$$

Equivalently, since $\eta_* = \eta_*(L)$, let $h \in \mathbb{R}$ be small such that $\eta = \eta_*(L+h) \in I$. Write limiting period $T_*(L) := T(0,\eta_*(L))$ and limiting mass $Q_*(L) := Q(0,\eta_*(L))$ as one-variable functions of $L > 0$. Thus, the limits (4.43) can also be written as

$$\begin{aligned} \lim_{h\to 0} \frac{T_*(L+h)-T_*(L)}{\eta_*(L+h)-\eta_*(L)} &= \lim_{h\to 0} \frac{T(0,\eta_*(L+h))-T(0,\eta_*(L))}{\eta_*(L+h)-\eta_*(L)} = \int_0^1 \partial_\eta \ell_0(\eta_*,t)\,dt, \\ \lim_{h\to 0} \frac{Q_*(L+h)-Q_*(L)}{\eta_*(L+h)-\eta_*(L)} &= \lim_{h\to 0} \frac{Q(0,\eta_*(L+h))-Q(0,\eta_*(L))}{\eta_*(L+h)-\eta_*(L)} = -\int_0^1 \partial_\eta q_0(\eta_*,t)\,dt. \end{aligned}$$

Since the period is fixed, it has $h = (L+h) - L = T_*(L+h) - T_*(L)$. Thus, the derivative $Q_*'(L)$ can be computed using definition, such that

$$\frac{dQ_*}{dL} = \lim_{h\to 0} \frac{Q_*(L+h)-Q_*(L)}{h} = \frac{\lim\limits_{h\to 0} \dfrac{Q_*(L+h)-Q_*(L)}{\eta_*(L+h)-\eta_*(L)}}{\lim\limits_{h\to 0} \dfrac{T_*(L+h)-T_*(L)}{\eta_*(L+h)-\eta_*(L)}} = \frac{-\int_0^1 \partial_\eta q_0(\eta_*,t)\,dt}{\int_0^1 \partial_\eta \ell_0(\eta_*,t)\,dt}. \tag{4.44}$$

Therefore, the integral quotient in (4.41) can be replaced by the exact derivative using (4.44), and the right derivative at $\epsilon = 0$ is computed as

$$\left.\frac{dQ}{d\epsilon}\right|_{\epsilon=0^+} = -\int_0^1 \partial_\epsilon q(0,\eta_*,t)\,dt - \frac{dQ_*}{dL}\int_0^1 \partial_\epsilon \ell(0,\eta_*,t)\,dt.$$

This completes the proof of this lemma. □

Now, we can prove monotonicity using the limiting mass formula for even waves (5.8) as following.

Proposition 4.3. *For fixed $L > 0$, the even wave mapping $\omega \mapsto Q$ is monotonically decreasing as $\omega \to 1^-$.*

Proof. Let $\epsilon = 1 - \omega \geq 0$ and fix $L > 0$ on the even branch $T(\epsilon,\eta(\epsilon)) = L$. By lemma 4.2, and the right derivative of Q can be computed as

$$\left.\frac{d}{d\epsilon} Q(\epsilon,\eta(\epsilon))\right|_{\epsilon=0^+} = -\int_0^1 \partial_\epsilon q(0,\eta_*,t)\,dt - \frac{dQ_*}{dL}\int_0^1 \partial_\epsilon \ell(0,\eta_*,t)\,dt. \tag{4.45}$$

and the integrals take the form

$$\int_0^1 \partial_\epsilon \ell(0,\eta_*,t)\,dt = -\frac{1}{2\sqrt{\eta_*}}K_1, \qquad -\int_0^1 \partial_\epsilon q(0,\eta_*,t)\,dt = \frac{1}{2\sqrt{\eta_*}}\left(K_1 \log\eta_* + K_2\right),$$

where we denote sign definite integrals

$$K_1 := \int_0^1 \frac{\log t}{(1-t)^{3/2}\sqrt{1-\eta_* t}}\, dt < 0, \qquad K_2 := \int_0^1 \frac{(\log t)^2}{(1-t)^{3/2}\sqrt{1-\eta_* t}}\, dt > 0.$$

The derivative (4.45) can be written as

$$\frac{d}{d\epsilon} Q(\epsilon, \eta(\epsilon))\bigg|_{\epsilon=0^+} = \frac{1}{2\sqrt{\eta_*}}\left[\left(\log \eta_* + \frac{dQ_*}{dL}\right) K_1 + K_2\right],$$

and we show it is positive by showing $P(L) := \log \eta_* + \frac{dQ_*}{dL}$ is negative. Use the explicit formula (5.10), we compute

$$P(L) = \log \eta_* + \frac{dQ_*}{dL} = L\left[1 + \tanh\left(\frac{L}{2}\right)\right] - 2\log(1 + e^L).$$

where $\lim_{L\to 0^+} P(L) = -2\log 2 < 0$ and $\lim_{L\to\infty} P(L) = 0$. Since the function is monotonically increasing with $P'(L) = \frac{1}{2} L \operatorname{sech}^2(\frac{L}{2}) > 0$, it follows that $P(L) < 0$ for all $L > 0$. Hence, the left derivative at $\omega = 1$ is

$$\frac{dQ}{d\omega}\bigg|_{\omega=1^-} = -\frac{d}{d\epsilon} Q(\epsilon, \eta(\epsilon))\bigg|_{\epsilon=0^+} < 0. \tag{4.46}$$

as desired. □

4.2 Endpoint monotonicity of odd waves

In this subsection, we will show that the mass functional is monotonically increasing near left endpoint $\omega = \Omega_L$ and monotonically decreasing near the right endpoint $\omega = 1$. The period and mass of odd waves can be written as

$$T = 4\int_0^M \frac{1}{\sqrt{2(\mathcal{E}_L - V)}}\, d\phi, \qquad Q = -4\int_0^M \frac{\log(1-\phi^2)}{\sqrt{2(\mathcal{E}_L - V)}}\, d\phi. \tag{4.47}$$

where the turning points $M(\omega) = \max_{x\in\mathbb{T}_L} \phi(x) \in (\sqrt{\omega}, 1)$ is the unique positive root of denominator (4.47) that can be written as

$$2(\mathcal{E}_L - V) = (2\mathcal{E}_L - 2\mathcal{E}_\omega + 1) - (1-\phi^2) + (1-\omega)\log(1-\phi^2) = 0, \tag{4.48}$$

Proposition 4.4. *For fixed $L > 0$, the odd wave mapping $\omega \mapsto Q$ is monotonically increasing near $\omega = \Omega_L^+$ where $\Omega_L = -\frac{4\pi^2}{L^2}$.*

Proof. By [34, Proposition 2.8], there exists $a_0 > 0$, such that for every $a \in (-a_0, a_0)$, there is an odd periodic solution $\phi \in H^2_{\mathrm{per,o}}$ represented as $\phi(x) = a\sin(kx) + \mathcal{O}(a^3), k = \frac{2\pi}{L}$. Use series expansion $-\log(1-\phi^2) = \phi^2 + \frac{1}{2}\phi^4 + \mathcal{O}(\phi^6)$ to compute the mass (1.4) as

$$Q_L(\omega) = \int_{\mathbb{T}_L} \phi^2\, dx + \mathcal{O}(a^4) = \frac{1}{2}a^2 L + \mathcal{O}(a^4) = \frac{2L}{3\left(1 + \frac{4\pi^2}{L^2}\right)}(\omega - \Omega_L) + \mathcal{O}\left((\omega - \Omega_L)^2\right) \tag{4.49}$$

so the positive first coefficient implies the right derivative $Q_L'(\Omega_L^+) > 0$ for all $L > 0$. □

For the right endpoint, the proof of the following lemma is similar to the even case except on root-finding formula (4.5) and (4.52). This leads to construction of different bounds and estimates, so we give a parallel proof here.

Lemma 4.5. *Fix $L > 0$ and let $\epsilon = 1 - \omega$ and $\gamma(\epsilon) = 2\mathcal{E}_L - 2\mathcal{E}_\omega + 1$ be the parameter determined by the fixed period odd branch $T(\epsilon, \gamma(\epsilon)) = L$. Then, the right derivative of mass at $\epsilon = 0$ can be computed as*

$$\frac{d}{d\epsilon} Q(\epsilon, \gamma(\epsilon))\bigg|_{\epsilon=0^+} = -2\int_0^1 \partial_\epsilon q(0, \gamma_*, t)\, dt - 2\frac{dQ_*}{dL}\int_0^1 \partial_\epsilon \ell(0, \gamma_*, t)\, dt. \tag{4.50}$$

where the right partial derivatives are computed using

$$\ell(\epsilon, \gamma, t) := \frac{1}{\sqrt{1-t}\sqrt{\gamma - t + \epsilon \log t}}, \quad q(\epsilon, \gamma, t) := \frac{\log t}{\sqrt{1-t}\sqrt{\gamma - t + \epsilon \log t}}, \tag{4.51}$$

and evaluated at the point $(0, \gamma_*, t)$ with $\gamma_* = \coth^2(\frac{L}{4})$.

Proof. Let $L > 0$ be fixed and $\epsilon = 1 - \omega \geq 0$ so that $\omega \to 1^-$ is equivalent to $\epsilon \to 0^+$. By change of variable $t = 1 - \phi^2 \in (0, 1)$ and $\gamma := 2\mathcal{E}_L - 2\mathcal{E}_\omega + 1 \in (1, \infty)$ in (4.48) and write the square root on denominator of (4.47) as

$$\Xi(\epsilon, \gamma, t) := 2(\mathcal{E}_L - V) = \gamma - t + \epsilon \log t, \qquad t \in (0, 1]. \tag{4.52}$$

Denote the new turning point as $\nu_\epsilon := 1 - M^2$ and mass and period in (4.47) can be rewritten as

$$T(\epsilon, \gamma) = 2\int_{\nu_\epsilon}^1 \ell(\epsilon, \gamma, t)\, dt, \qquad Q(\epsilon, \gamma) = -2\int_{\nu_\epsilon}^1 q(\epsilon, \gamma, t)\, dt, \qquad \epsilon > 0, \tag{4.53}$$

where the integrands (4.51) are represented as

$$\ell(\epsilon, \gamma, t) = \frac{1}{\sqrt{1-t}\sqrt{\Xi(\epsilon, \gamma, t)}}, \qquad q(\epsilon, \gamma, t) = \frac{\log t}{\sqrt{1-t}\sqrt{\Xi(\epsilon, \gamma, t)}}. \tag{4.54}$$

Note that functions (4.54) are integrable on $(\nu_\epsilon, 1)$ since $\ell, q \sim \mathcal{O}\left((t - \nu_\epsilon)^{-1/2}\right)$ as $t \to \nu_\epsilon^+$ and $\ell \sim \mathcal{O}\left((1-t)^{-1/2}\right), q \sim \mathcal{O}\left((1-t)^{1/2}\right)$ as $t \to 1^-$. At $\epsilon = 0$, define $\nu_0 := 0$ and exact solution (5.3) implies $\gamma_* := \gamma(0) = \coth^2(\frac{L}{4}) > 1$, so they can be considered as the right limits

$$\lim_{\epsilon \to 0^+} \nu_\epsilon = \nu_0, \qquad \lim_{\epsilon \to 0^+} \gamma = \gamma_*.$$

We aim to compute right derivative using limit definition as $\epsilon \to 0^+$ and $\gamma(\epsilon) \to \gamma_*$. Note that $\gamma \in (1, \infty)$ is an independent variable in (4.54), so we define the compact set J, such that

$$J := \left[\frac{\gamma_* + 1}{2}, \frac{3\gamma_*}{2}\right] \subset (1, \infty), \qquad \gamma_* = \coth^2\left(\frac{L}{4}\right) \in J, \quad L > 0. \tag{4.55}$$

For sufficiently small $\epsilon > 0$, we can construct uniform estimates for $\gamma \in J$. Define the integrals (4.53) and integrands (4.54) with $\epsilon = 0$ and $\gamma \in J$, such that

$$T(0,\gamma) := 2\int_0^1 \ell(0,\gamma,t)\,dt, \qquad Q(0,\gamma) := -2\int_0^1 q(0,\gamma,t)\,dt, \qquad \epsilon = 0, \tag{4.56}$$

where

$$\ell(0,\gamma,t) := \frac{1}{\sqrt{1-t}\sqrt{\gamma-t}}, \qquad q(0,\gamma,t) := \frac{\log t}{\sqrt{1-t}\sqrt{\gamma-t}}. \tag{4.57}$$

Since $\gamma > 1$, the factor $(\gamma - t)^{-1/2}$ is bounded on $t \in (0,1)$, so it has integrable singularities $(1-t)^{-1/2}$ for $\ell(0,\gamma,t)$ near $t = 1$ and $\log t$ for $q(0,\gamma,t)$ near $t = 0$, i.e. $\ell(0,\gamma,t), q(0,\gamma,t) \in L^1_t(0,1)$. For $\epsilon \geq 0$ and $\gamma \in J$, use (4.53) and (4.56) to write the difference quotient of period and mass as

$$\frac{T(\epsilon,\gamma) - T(0,\gamma_*)}{\epsilon} = 2\int_{\nu_\epsilon}^1 \frac{\ell(\epsilon,\gamma,t) - \ell(0,\gamma_*,t)}{\epsilon}\,dt - \frac{2}{\epsilon}\int_0^{\nu_\epsilon} \ell(0,\gamma_*,t)\,dt \tag{4.58}$$

$$\frac{Q(\epsilon,\gamma) - Q(0,\gamma_*)}{\epsilon} = -2\int_{\nu_\epsilon}^1 \frac{q(\epsilon,\gamma,t) - q(0,\gamma_*,t)}{\epsilon}\,dt + \frac{2}{\epsilon}\int_0^{\nu_\epsilon} q(0,\gamma_*,t)\,dt \tag{4.59}$$

and we would like to compute their limit as $\epsilon \to 0^+$.

To start, we show that the second integrals in (4.58) and (4.59) vanish as

$$\lim_{\epsilon\to 0^+} \frac{2}{\epsilon}\int_0^{\nu_\epsilon} \ell(0,\gamma_*,t)\,dt = 0, \qquad \lim_{\epsilon\to 0^+} \frac{2}{\epsilon}\int_0^{\nu_\epsilon} q(0,\gamma_*,t)\,dt = 0, \tag{4.60}$$

since ν_ϵ is exponentially small. For $\epsilon > 0$, the function (4.52) has $\lim_{t\to 0^+} \Xi = -\infty$ and $\Xi(\epsilon,\gamma,1) = \gamma - 1 > 0$. Since $\partial_t \Xi = 0$ at $t = \epsilon$ and $\partial_t^2 \Xi = -\epsilon t^{-2} < 0$, the function (4.52) has a unique maximum at $t = \epsilon$. Thus, for sufficiently small $\epsilon > 0$, the root has $\nu_\epsilon < \frac{1}{2}$ and it implies

$$\nu_\epsilon = \exp\left[-\frac{\gamma - \nu_\epsilon}{\epsilon}\right] \leq e^{-\gamma/(2\epsilon)}, \tag{4.61}$$

which can be used to construct bounds. Since $\nu_\epsilon < \frac{1}{2}$ and $\gamma_* > 1$, the denominators of (4.57) has $\sqrt{(1-\nu_\epsilon)(\gamma_* - \nu_\epsilon)} > \frac{1}{2}$ and the period can be estimated as

$$\frac{2}{\epsilon}\int_0^{\nu_\epsilon} \ell(0,\gamma_*,t)\,dt \leq \left(\frac{\nu_\epsilon}{\epsilon}\right)\frac{2}{\sqrt{1-\nu_\epsilon}\sqrt{\gamma_* - \nu_\epsilon}} \leq \frac{4\nu_\epsilon}{\epsilon}. \tag{4.62}$$

Similarly, since $\sqrt{(1-\nu_\epsilon)(\gamma_* - \nu_\epsilon)} > \frac{1}{2}$, the mass can be estimated as

$$\frac{2}{\epsilon}\int_0^{\nu_\epsilon} |q(0,\gamma_*,t)|\,dt \leq \frac{2}{\epsilon}\int_0^{\nu_\epsilon} 2|\log t|\,dt \leq \frac{4\nu_\epsilon}{\epsilon}(1 + |\log \nu_\epsilon|). \tag{4.63}$$

As $\epsilon \to 0^+$, the inequality (4.61) implies the turning point $\nu_\epsilon \to 0^+$ exponentially fast, and thus bounds (4.62), (4.63) tends to zero and this proves the limits in (4.60).

Then, we only need to analyze the first integrals in (4.58) and (4.59). By the fundamental theorem of calculus, let $s \in [0,1]$ be a parameter so their integrands can be written as

$$\begin{aligned}\frac{\ell(\epsilon,\gamma,t)-\ell(0,\gamma_*,t)}{\epsilon} &= \frac{\ell(\epsilon,\gamma,t)-\ell(0,\gamma,t)}{\epsilon} + \frac{\ell(0,\gamma,t)-\ell(0,\gamma_*,t)}{\epsilon} \\ &= \int_0^1 \partial_\epsilon \ell(s\epsilon,\gamma,t)\,ds + \frac{\gamma-\gamma_*}{\epsilon}\int_0^1 \partial_\gamma \ell_0(\gamma_*+s(\gamma-\gamma_*),t)\,ds,\end{aligned} \tag{4.64}$$

and

$$\begin{aligned}\frac{q(\epsilon,\gamma,t)-q(0,\gamma_*,t)}{\epsilon} &= \frac{q(\epsilon,\gamma,t)-q(0,\gamma,t)}{\epsilon} + \frac{q(0,\gamma,t)-q(0,\gamma_*,t)}{\epsilon} \\ &= \int_0^1 \partial_\epsilon q(s\epsilon,\gamma,t)\,ds + \frac{\gamma-\gamma_*}{\epsilon}\int_0^1 \partial_\gamma q_0(\gamma_*+s(\gamma-\gamma_*),t)\,ds.\end{aligned} \tag{4.65}$$

The partial derivatives of three-variable functions $\ell(\epsilon,\gamma,t)$ and $q(\epsilon,\gamma,t)$ respect to $\epsilon \in (0,1)$ can be computed as

$$\partial_\epsilon \ell(\epsilon,\gamma,t) = -\frac{\log t}{2\sqrt{1-t}\,[\Xi(\epsilon,\gamma,t)]^{3/2}}, \quad \partial_\epsilon q(\epsilon,\gamma,t) = -\frac{(\log t)^2}{2\sqrt{1-t}\,[\Xi(\epsilon,\gamma,t)]^{3/2}} \tag{4.66}$$

and they are evaluated at point $(s\epsilon,\gamma,t)$ as the first integrands in (4.64) and (4.65). At $\epsilon = 0$, the partial derivatives of two-variable functions $\ell_0(\gamma,t) := \ell(0,\gamma,t)$ and $q_0(\gamma,t) := q(0,\gamma,t)$ respect to $\gamma \in (1,\infty)$ can be computed as

$$\partial_\gamma \ell_0(\gamma,t) = -\frac{1}{2\sqrt{1-t}\,(\gamma-t)^{3/2}}, \quad \partial_\gamma q_0(\gamma,t) = -\frac{\log t}{2\sqrt{1-t}\,(\gamma-t)^{3/2}} \tag{4.67}$$

and they are evaluated at point $(\gamma_*+s(\gamma-\gamma_*),t)$ to be the second integrands in (4.64) and (4.65).

The first quotients from (4.64) and (4.65) in integral representation reads

$$\frac{\ell(\epsilon,\gamma,t)-\ell(0,\gamma,t)}{\epsilon} = \int_0^1 \partial_\epsilon \ell(s\epsilon,\gamma,t)\,ds, \quad \frac{q(\epsilon,\gamma,t)-q(0,\gamma,t)}{\epsilon} = \int_0^1 \partial_\epsilon q(s\epsilon,\gamma,t)\,ds, \tag{4.68}$$

which are then substituted into integrals (4.58) and (4.59). Let $\epsilon > 0$ be sufficiently small so it satisfies

$$\gamma \in J, \qquad \nu_\epsilon < e^{-1/\sqrt{\epsilon}} < \frac{1}{2}\epsilon, \qquad \epsilon < \frac{\gamma^2}{16}, \tag{4.69}$$

and we decompose the region of integration as

$$(\nu_\epsilon,1) = (\nu_\epsilon,t_\epsilon) \cup \left(t_\epsilon,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right), \qquad t_\epsilon := e^{-\frac{1}{\sqrt{\epsilon}}}. \tag{4.70}$$

We prove that for each $\gamma \in J$, the quotients (4.68) as integrands in (4.58) and (4.59) have the limits

$$\begin{aligned} \lim_{\epsilon \to 0^+} \int_{\nu_\epsilon}^1 \frac{\ell(\epsilon,\gamma,t) - \ell(0,\gamma,t)}{\epsilon}\, dt &= \int_0^1 \partial_\epsilon \ell(0,\gamma,t)\, dt, \\ \lim_{\epsilon \to 0^+} \int_{\nu_\epsilon}^1 \frac{q(\epsilon,\gamma,t) - q(0,\gamma,t)}{\epsilon}\, dt &= \int_0^1 \partial_\epsilon q(0,\gamma,t)\, dt. \end{aligned} \tag{4.71}$$

We separate $(\nu_\epsilon, 1)$ into three regions as (4.70) and analyze each region.

(i) Region $(\nu_\epsilon, t_\epsilon)$:
We show that the integrals vanish uniformly for $\gamma \in J$, such that

$$\lim_{\epsilon \to 0^+} \int_{\nu_\epsilon}^{t_\epsilon} \frac{\ell(\epsilon,\gamma,t) - \ell(0,\gamma,t)}{\epsilon}\, dt = 0, \qquad \lim_{\epsilon \to 0^+} \int_{\nu_\epsilon}^{t_\epsilon} \frac{q(\epsilon,\gamma,t) - q(0,\gamma,t)}{\epsilon}\, dt = 0. \tag{4.72}$$

By triangle inequality, the integrals can be estimated as

$$\begin{aligned} \left| \int_{\nu_\epsilon}^{t_\epsilon} \frac{\ell(\epsilon,\gamma,t) - \ell(0,\gamma,t)}{\epsilon}\, dt \right| &\le \frac{1}{\epsilon} \int_{\nu_\epsilon}^{t_\epsilon} \left(|\ell(\epsilon,\gamma,t)| + |\ell(0,\gamma,t)| \right) dt, \\ \left| \int_{\nu_\epsilon}^{t_\epsilon} \frac{q(\epsilon,\gamma,t) - q(0,\gamma,t)}{\epsilon}\, dt \right| &\le \frac{1}{\epsilon} \int_{\nu_\epsilon}^{t_\epsilon} \left(|q(\epsilon,\gamma,t)| + |q(0,\gamma,t)| \right) dt, \end{aligned} \tag{4.73}$$

and we use (4.54) to construct the bounds.

For the term $|\ell(0,\gamma,t)|$, since the mapping $t \mapsto (\sqrt{1-t}\sqrt{\gamma - t})^{-1}$ is monotonically increasing on $(0,1)$, the maximum over the interval $[\nu_\epsilon, t_\epsilon]$ is attained at $t = t_\epsilon$, and it follows that

$$\frac{1}{\epsilon} \int_{\nu_\epsilon}^{t_\epsilon} |\ell(0,\gamma,t)|\, dt \le \frac{1}{\sqrt{1-t_\epsilon}\sqrt{\gamma - t_\epsilon}} \left(\frac{t_\epsilon}{\epsilon} \right) =: D_{\ell,0} \left(\frac{t_\epsilon}{\epsilon} \right), \quad D_{\ell,0}(\epsilon,\gamma) > 0. \tag{4.74}$$

In addition, since the mapping $t \mapsto |\log t|$ is monotonically decreasing on $(0,1)$, it has $\sup_{t \in [\nu_\epsilon, t_\epsilon]} |\log t| = |\log \nu_\epsilon| = \frac{\gamma - \nu_\epsilon}{\epsilon}$, so the term $|q(0,\gamma,t)|$ has the bound

$$\frac{1}{\epsilon} \int_{\nu_\epsilon}^{t_\epsilon} |q(0,\gamma,t)|\, dt \le \frac{\gamma - \nu_\epsilon}{\sqrt{1-t_\epsilon}\sqrt{\gamma - t_\epsilon}} \left(\frac{t_\epsilon}{\epsilon^2} \right) =: D_{q,0} \left(\frac{t_\epsilon}{\epsilon^2} \right), \quad D_{q,0}(\epsilon,\gamma) > 0 \tag{4.75}$$

For terms $|\ell(\epsilon,\gamma,t)|$ and $|q(\epsilon,\gamma,t)|$, the $\sqrt{\Xi(\epsilon,\gamma,t)}$ in the denominators of (4.54) stays positive on $t \in (\nu_\epsilon, t_\epsilon)$ and vanishes at $t = \nu_\epsilon$. Since $t < t_\epsilon < \frac{\epsilon}{2}$ and small $\epsilon > 0$ is chosen to satisfy (4.69), it has $-1 \ge -\frac{\epsilon}{2t}$, so that $\partial_t \Xi = -1 + \frac{\epsilon}{t} \ge \frac{\epsilon}{2t}$. Use $\Xi(\epsilon,\gamma,\nu_\epsilon) = 0$ and the elementary inequality $\log x \ge \frac{x-1}{x}$ for $x = \frac{t}{\nu_\epsilon} > 0$, so the fundamental theorem of calculus gives

$$\Xi(\epsilon,\gamma,t) = \int_{\nu_\epsilon}^t \partial_r \Xi(\epsilon,\gamma,r)\, dr \ge \frac{\epsilon}{2} \int_{\nu_\epsilon}^t \frac{1}{r}\, dr = \frac{\epsilon}{2} \log\left(\frac{t}{\nu_\epsilon} \right) \ge \left(\frac{\epsilon}{2} \right) \frac{t - \nu_\epsilon}{t}$$

i.e. $\Xi^{-1/2} \leq \sqrt{\frac{2t}{\epsilon(t-\nu_\epsilon)}}$ for $t \in (\nu_\epsilon, t_\epsilon)$. Since the mapping $t \mapsto \frac{1}{\sqrt{1-t}}$ is increasing on $(0,1)$, the maximum over the interval $[\nu_\epsilon, t_\epsilon]$ is attained at $t = t_\epsilon$, and thus $|\ell(\epsilon,\gamma,t)|$ has the bound

$$
\begin{aligned}
\frac{1}{\epsilon}\int_{\nu_\epsilon}^{t_\epsilon} |\ell(\epsilon,\gamma,t)|\,dt &\leq \frac{1}{\epsilon^{3/2}}\sqrt{\frac{2t_\epsilon}{1-t_\epsilon}}\int_{\nu_\epsilon}^{t_\epsilon}\frac{1}{\sqrt{t-\nu_\epsilon}}\,dt \\
&= \frac{2\sqrt{2}}{\sqrt{1-t_\epsilon}}\sqrt{1-\frac{\nu_\epsilon}{t_\epsilon}}\left(\frac{t_\epsilon}{\epsilon^{3/2}}\right) =: D_{\ell,1}\left(\frac{t_\epsilon}{\epsilon^{3/2}}\right), \quad D_{\ell,1}(\epsilon,\eta) > 0. \qquad (4.76)
\end{aligned}
$$

By analogy to (4.75) and (4.76), with $|\log t| \leq |\log \nu_\epsilon| = \frac{\gamma-\nu_\epsilon}{\epsilon}$, the term $|q(\epsilon,\gamma,t)|$ can be bounded as

$$
\frac{1}{\epsilon}\int_{\nu_\epsilon}^{t_\epsilon} |q(\epsilon,\gamma,t)|\,dt \leq \frac{2\sqrt{2}(\gamma-\nu_\epsilon)}{\sqrt{1-t_\epsilon}}\sqrt{1-\frac{\nu_\epsilon}{t_\epsilon}}\left(\frac{t_\epsilon}{\epsilon^{5/2}}\right) =: D_{q,1}\left(\frac{t_\epsilon}{\epsilon^{5/2}}\right). \qquad (4.77)
$$

where $D_{q,1}(\epsilon,\eta) > 0$ is some constant.

In conclusion, for some small $\epsilon_0 > 0$, we can take the supremum of coefficients in (4.74),(4.75),(4.76), and (4.77) to represent as a positive constant $D > 0$ that is uniformly bounded for $\epsilon \in (0,\epsilon_0)$ and $\gamma \in J$, such that

$$
D := \sup_{\substack{\epsilon\in(0,\epsilon_0)\\ \gamma\in J}} \max\left\{D_{\ell,0}(\epsilon,\gamma), D_{q,0}(\epsilon,\gamma), D_{\ell,1}(\epsilon,\gamma), D_{q,1}(\epsilon,\gamma)\right\}.
$$

where each terms in (4.73) can be further estimated as

$$
\begin{aligned}
&\frac{1}{\epsilon}\int_{\nu_\epsilon}^{t_\epsilon} |\ell(0,\gamma,t)|\,dt \leq D\left(\frac{t_\epsilon}{\epsilon}\right), \qquad &&\frac{1}{\epsilon}\int_{\nu_\epsilon}^{t_\epsilon} |\ell(\epsilon,\gamma,t)|\,dt \leq D\left(\frac{t_\epsilon}{\epsilon^{3/2}}\right), \\
&\frac{1}{\epsilon}\int_{\nu_\epsilon}^{t_\epsilon} |q(0,\gamma,t)|\,dt \leq D\left(\frac{t_\epsilon}{\epsilon^{2}}\right), \qquad &&\frac{1}{\epsilon}\int_{\nu_\epsilon}^{t_\epsilon} |q(\epsilon,\gamma,t)|\,dt \leq D\left(\frac{t_\epsilon}{\epsilon^{5/2}}\right).
\end{aligned}
$$

As $\epsilon \to 0^+$, since $\frac{t_\epsilon}{\epsilon^a} \to 0$ for any finite $a > 0$, the limits in (4.72) are justified.

(ii) Region $(t_\epsilon, \frac{1}{2})$:

We show that the quotients $\frac{\ell(\epsilon,\gamma,t)-\ell(0,\gamma,t)}{\epsilon}\chi_{[t_\epsilon,\frac{1}{2}]}(t)$, $\frac{q(\epsilon,\gamma,t)-q(0,\gamma,t)}{\epsilon}\chi_{[t_\epsilon,\frac{1}{2}]}(t) \in L^1_t(0,\frac{1}{2})$ for all $\gamma \in J$, and they have the limits

$$
\begin{aligned}
\lim_{\epsilon\to 0^+}\int_{t_\epsilon}^{1/2}\frac{\ell(\epsilon,\gamma,t)-\ell(0,\gamma,t)}{\epsilon}\,dt &= \int_0^{1/2}\partial_\epsilon \ell(0,\gamma,t)\,dt, \\
\lim_{\epsilon\to 0^+}\int_{t_\epsilon}^{1/2}\frac{q(\epsilon,\gamma,t)-q(0,\gamma,t)}{\epsilon}\,dt &= \int_0^{1/2}\partial_\epsilon q(0,\gamma,t)\,dt.
\end{aligned} \qquad (4.78)
$$

From (4.64) and (4.65), the quotients have the integral representation (4.68), so it suffices to show that $\chi_{[t_\epsilon,\frac{1}{2}]}(t)\partial_\epsilon\ell(s\epsilon,\gamma,t), \chi_{[t_\epsilon,\frac{1}{2}]}(t)\partial_\epsilon q(s\epsilon,\gamma,t) \in L^1_t(0,\frac{1}{2})$.

Since $t_\epsilon < t < \frac{1}{2}$ and $\gamma > 1$, we have $\epsilon|\log t| < \epsilon|\log t_\epsilon| \le \sqrt{\epsilon}$, and the function $\Xi(s\epsilon, \gamma, t), s \in [0,1]$ has the lower bound

$$\Xi(s\epsilon, \gamma, t) = \gamma - t + s\epsilon \log t \ge \frac{1}{2} - \epsilon|\log t| \ge \frac{1}{2}\gamma - \sqrt{\epsilon}.$$

By (4.69), the small $\epsilon > 0$ is chosen to satisfy $\sqrt{\epsilon} < \frac{\gamma}{4}$, which gives a strictly positive bound $\Xi(s\epsilon, \gamma, t) \ge \frac{\gamma}{4} > 0$. The partial derivatives are computed in (4.66), with $\sqrt{1-t} \ge \frac{1}{\sqrt{2}}$ and $[\Xi(s\epsilon, \gamma, t)]^{-3/2} \le \frac{8}{\gamma^{3/2}}$, they are evaluated at the point $(s\epsilon, \gamma, t)$ and estimated as

$$\begin{aligned}
|\partial_\epsilon \ell(s\epsilon, \gamma, t)| &= \frac{|\log t|}{2\sqrt{1-t}\,[\Xi(s\epsilon, \gamma, t)]^{3/2}} \le \frac{4\sqrt{2}}{\gamma^{3/2}}|\log t|, \\
|\partial_\epsilon q(s\epsilon, \gamma, t)| &= \frac{(\log t)^2}{2\sqrt{1-t}\,[\Xi(s\epsilon, \gamma, t)]^{3/2}} \le \frac{4\sqrt{2}}{\gamma^{3/2}}|\log t|^2.
\end{aligned}$$

Since $|\log t|, |\log t|^2 \in L^1_t(0, \frac{1}{2})$, we can define the majorant $h_1 \in L^1_t(0, \frac{1}{2})$, such that

$$h_1(t) = D_1 \left(|\log t| + |\log t|^2\right), \quad D_1 := \sup_{\gamma \in J} \left(\frac{4\sqrt{2}}{\gamma^{3/2}}\right) < \infty$$

where $|\partial_\epsilon \ell(s\epsilon, \gamma, t)| \le h_1$ and $|\partial_\epsilon q(s\epsilon, \gamma, t)| \le h_1$ for all $t \in (0, \frac{1}{2})$, $s \in [0,1]$ and $\gamma \in J$. This directly implies that $\partial_\epsilon \ell(s\epsilon, \gamma, t), \partial_\epsilon q(s\epsilon, \gamma, t) \in L^1_s(0,1)$ for each fixed $t \in (t_\epsilon, \frac{1}{2})$ since h_1 is independent of $s \in [0,1]$. As $\epsilon \to 0^+$, for each fixed $t \in (0, \frac{1}{2})$, $\gamma \in J$ and $s \in [0,1]$, it has pointwise convergence $\Xi(s\epsilon, \gamma, t) \to \Xi(0, \gamma, t)$ a.e. so that $\partial_\epsilon \ell(s\epsilon, \gamma, t) \to \partial_\epsilon \ell(0, \gamma, t)$ and $\partial_\epsilon q(s\epsilon, \gamma, t) \to \partial_\epsilon q(0, \gamma, t)$ pointwise a.e. by (4.66). Since $\chi_{[t_\epsilon, \frac{1}{2}]}(t) \to 1$ pointwise a.e. as $\epsilon \to 0^+$ and $\partial_\epsilon \ell(s\epsilon, \gamma, t), \partial_q \ell(s\epsilon, \gamma, t) \in L^1_s(0,1)$, by dominated convergence theorem in $s \in (0,1)$, it has

$$\begin{aligned}
&\chi_{\left[t_\epsilon, \frac{1}{2}\right]}(t) \int_0^1 \partial_\epsilon \ell(s\epsilon, \gamma, t)\, ds \to \partial_\epsilon \ell(0, \gamma, t), \\
&\chi_{\left[t_\epsilon, \frac{1}{2}\right]}(t) \int_0^1 \partial_\epsilon q(s\epsilon, \gamma, t)\, ds \to \partial_\epsilon q(0, \gamma, t),
\end{aligned} \tag{4.79}$$

pointwise a.e. as $\epsilon \to 0^+$ and admits the majorant

$$\left|\chi_{\left[t_\epsilon, \frac{1}{2}\right]}(t) \int_0^1 \partial_\epsilon \ell(s\epsilon, \gamma, t)\, ds\right| \le h_1, \quad \left|\chi_{\left[t_\epsilon, \frac{1}{2}\right]}(t) \int_0^1 \partial_\epsilon q(s\epsilon, \gamma, t)\, ds\right| \le h_1.$$

Thus, use (4.68) to replace the integrals in (4.79) by the quotients and apply dominated convergence theorem again on $t \in (0, \frac{1}{2})$, so it follows that

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_{t_\epsilon}^{1/2}\frac{\ell(\epsilon,\gamma,t)-\ell(0,\eta,t)}{\epsilon}\,dt &= \lim_{\epsilon\to 0^+}\int_0^{1/2}\frac{\ell(\epsilon,\gamma,t)-\ell(0,\eta,t)}{\epsilon}\chi_{\left[t_\epsilon,\frac{1}{2}\right]}(t)\,dt\\ &= \int_0^{1/2}\partial_\epsilon\ell(0,\gamma,t)\,dt,\\ \lim_{\epsilon\to 0^+}\int_{t_\epsilon}^{1/2}\frac{q(\epsilon,\gamma,t)-q(0,\gamma,t)}{\epsilon}\,dt &= \lim_{\epsilon\to 0^+}\int_0^{1/2}\frac{q(\epsilon,\gamma,t)-q(0,\gamma,t)}{\epsilon}\chi_{\left[t_\epsilon,\frac{1}{2}\right]}(t)\,dt\\ &= \int_0^{1/2}\partial_\epsilon q(0,\gamma,t)\,dt.\end{aligned}$$

This proves the limits in (4.78).

(iii) Region $(\frac{1}{2}, 1)$:

We show that the quotient $\frac{\ell(\epsilon,\gamma,t)-\ell(0,\gamma,t)}{\epsilon}, \frac{q(\epsilon,\gamma,t)-q(0,\eta,t)}{\epsilon} \in L^1(\frac{1}{2}, 1)$ and they satisfy the limits

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_{1/2}^{1}\frac{\ell(\epsilon,\gamma,t)-\ell(0,\eta,t)}{\epsilon}\,dt &= \int_{1/2}^{1}\partial_\epsilon\ell(0,\gamma,t)\,dt,\\ \lim_{\epsilon\to 0^+}\int_{1/2}^{1}\frac{q(\epsilon,\gamma,t)-q(0,\eta,t)}{\epsilon}\,dt &= \int_{1/2}^{1}\partial_\epsilon q(0,\gamma,t)\,dt.\end{aligned} \tag{4.80}$$

By the fundamental theorem of calculus, the quotients admits the integral representation (4.68), so it suffices to show that $\partial_\epsilon\ell(s\epsilon,\gamma,t), \partial_\epsilon q(s\epsilon,\gamma,t) \in L^1_t(\frac{1}{2}, 1)$ uniformly for $s \in [0, 1]$ and $\gamma \in J$.

Since $t \in (\frac{1}{2}, 1)$ and $\gamma > 1$, it has $\gamma - t > 1 - t$ and $|\log t| = \int_t^1 r^{-1}\,dr \le 2(1-t)$. For every $s \in [0, 1]$ and sufficiently small $\epsilon > 0$ chosen from (4.69), it has $2\epsilon < \frac{\gamma^2}{8} < \frac{\gamma}{2}$, which gives the strictly positive lower bound

$$\Xi(s\epsilon,\gamma,t) \ge \gamma - t - \epsilon|\log t| \ge (\gamma - 2\epsilon)(1-t) \ge \frac{\gamma}{2}(1-t) > 0. \tag{4.81}$$

The partial derivatives are computed in (4.66), by using $\gamma - t > \gamma - \frac{1}{2}$, $|\log t| \le \log 2$, $\frac{|\log t|}{1-t} \le 2$ and lower bound (4.81), they can be estimated as

$$\begin{aligned}|\partial_\epsilon\ell(s\epsilon,\gamma,t)| &= \frac{|\log t|}{2\sqrt{\gamma-t}\,[\Xi(s\epsilon,\gamma,t)]^{3/2}} \le \frac{4}{\sqrt{\gamma^3(2\gamma-1)}\sqrt{1-t}},\\ |\partial_\epsilon q(s\epsilon,\gamma,t)| &= \frac{|\log t|^2}{2\sqrt{\gamma-t}\,[\Xi(s\epsilon,\gamma,t)]^{3/2}} \le \frac{4\log 2}{\sqrt{\gamma^3(2\gamma-1)}\sqrt{1-t}}.\end{aligned}$$

Since $(1-t)^{-1/2} \in L^1_t(\frac{1}{2}, 1)$, we can define the majorant $h_2 \in L^1_t(\frac{1}{2}, 1)$, such that

$$h_2(t) = \frac{D_2}{\sqrt{1-t}}, \qquad D_2 := \sup_{\gamma\in J}\left(\frac{4\log 2}{\sqrt{\gamma^3(2\gamma-1)}}\right) < \infty$$

where $|\partial_\epsilon \ell(s\epsilon,\gamma,t)| \leq h_2$ and $|\partial_\epsilon q(s\epsilon,\gamma,t)| \leq h_2$ for all $t \in (\frac{1}{2},1)$, $s \in [0,1]$ and $\gamma \in J$. This directly implies that $\partial_\epsilon \ell(s\epsilon,\gamma,t), \partial_\epsilon q(s\epsilon,\gamma,t) \in L^1_s(0,1)$ for each fixed $t \in (\frac{1}{2},1)$ since h_2 is independent of $s \in [0,1]$. As $\epsilon \to 0^+$, for each fixed $t \in (\frac{1}{2},1)$, $\gamma \in J$ and $s \in [0,1]$, it has pointwise convergence $\Xi(s\epsilon,\gamma,t) \to \Xi(0,\eta,t)$ a.e. so that $\partial_\epsilon \ell(s\epsilon,\gamma,t) \to \partial_\epsilon \ell(0,\gamma,t)$ and $\partial_\epsilon q(s\epsilon,\gamma,t) \to \partial_\epsilon q(0,\gamma,t)$ pointwise a.e. by (4.66). With $\partial_\epsilon \ell(s\epsilon,\gamma,t), \partial_\epsilon q(s\epsilon,\gamma,t) \in L^1_s(0,1)$, by dominated convergence theorem in $s \in (0,1)$, it follows that

$$\int_0^1 \partial_\epsilon \ell(s\epsilon,\gamma,t)\,ds \to \partial_\epsilon \ell(0,\gamma,t), \qquad \int_0^1 \partial_\epsilon q(s\epsilon,\gamma,t)\,ds \to \partial_\epsilon q(0,\gamma,t) \tag{4.82}$$

pointwise a.e. on $t \in (\frac{1}{2},1)$, and moreover

$$\left|\int_0^1 \partial_\epsilon \ell(s\epsilon,\gamma,t)\,ds\right| \leq h_2, \qquad \left|\int_0^1 \partial_\epsilon q(s\epsilon,\gamma,t)\,ds\right| \leq h_2.$$

Thus, use (4.68) to replace the integrals in (4.82) by the quotients and apply dominated convergence theorem again on $t \in (\frac{1}{2},1)$, so it proves the limits (4.80).

To summarize, the limits (4.72), (4.78) and (4.80) are uniform for $\gamma \in J$ where J is chosen in (4.55). As $\epsilon \to 0^+$, since $\gamma \to \gamma_*$ and $\gamma_* \in J$, it has $\partial_\epsilon \ell(0,\gamma,t) \to \partial_\epsilon \ell(0,\gamma_*,t)$ and $\partial_\epsilon q(0,\gamma,t) \to \partial_\epsilon q(0,\gamma_*,t)$ after passing the limit inside each integral by dominated convergence theorem, so that (4.71) becomes

$$\begin{aligned}\lim_{\epsilon\to 0^+} \int_{\nu_\epsilon}^1 \frac{\ell(\epsilon,\gamma,t) - \ell(0,\gamma,t)}{\epsilon}\,dt &= \int_0^1 \partial_\epsilon \ell(0,\gamma_*,t)\,dt,\\ \lim_{\epsilon\to 0^+} \int_{\nu_\epsilon}^1 \frac{q(\epsilon,\gamma,t) - q(0,\gamma,t)}{\epsilon}\,dt &= \int_0^1 \partial_\epsilon q(0,\gamma_*,t)\,dt.\end{aligned} \tag{4.83}$$

This finishes the analysis of the first quotients of (4.64) and (4.65) in (4.68).

It remains for us to consider the second quotients from (4.64) and (4.65) with integral representation

$$\begin{aligned}\frac{\ell(0,\gamma,t) - \ell(0,\gamma_*,t)}{\epsilon} &= \frac{\gamma-\gamma_*}{\epsilon} \int_0^1 \partial_\gamma \ell_0(\xi_s,t)\,ds,\\ \frac{q(0,\gamma,t) - q(0,\gamma_*,t)}{\epsilon} &= \frac{\gamma-\gamma_*}{\epsilon} \int_0^1 \partial_\gamma q_0(\xi_s,t)\,ds.\end{aligned} \tag{4.84}$$

where $\xi_s := \gamma_* + s(\gamma-\gamma_*) \in J$ and the partial derivatives in integrands are computed as (4.67) and evaluated at the point (ξ_s,t). We compute the limits $\epsilon \to 0^+$ of integrals in (4.84). For every $t \in (0,1)$, $s \in [0,1]$, $\xi_s \in J$ and sufficiently small $\epsilon > 0$ satisfying (4.69), it has $\xi_s - t \geq \xi_s - 1$. Denote the positive constant $D_3 := \frac{1}{2}\sup_{\xi_s\in J}(\xi_s-1)^{-3/2} < \infty$ so the derivatives can be estimated as

$$\begin{aligned}|\partial_\gamma \ell_0(\xi_s,t)| &= \frac{1}{2\sqrt{1-t}(\xi_s-t)^{3/2}} \leq \frac{D_3}{\sqrt{1-t}} =: h_3(t),\\ |\partial_\gamma q_0(\xi_s,t)| &= \frac{|\log t|}{2\sqrt{1-t}(\xi_s-t)^{3/2}} \leq \frac{D_3|\log t|}{\sqrt{1-t}} =: h_4(t).\end{aligned} \tag{4.85}$$

For each fixed $t \in (0,1)$, since the majorant are independent of $s \in (0,1)$, it has $h_3, h_4 \in L^1_s(0,1)$ so that $\partial_\gamma \ell_0(\xi_s, t), \partial_\gamma q_0(\xi_s, t) \in L^1_s(0,1)$. As $\epsilon \to 0^+$, we have $\nu_\epsilon \to 0^+$ and $\xi_s \to \gamma_*$ as $\epsilon \to 0^+$ so that $\partial_\gamma \ell_0(\xi_s, t) \to \partial_\gamma \ell_0(\gamma_*, t)$ and $\partial_\gamma q_0(\xi_s, t) \to \partial_\gamma q_0(\gamma_*, t)$ pointwise a.e. as $\epsilon \to 0^+$. Since $\chi_{[\nu_\epsilon,1]}(t) \to 1$ pointwise a.e. on $t \in (0,1)$, by dominated convergence theorem on $s \in (0,1)$, it has

$$\chi_{[\nu_\epsilon,1]}(t) \int_0^1 \partial_\gamma \ell_0(\xi_s, t)\, ds \to \partial_\gamma \ell_0(\gamma_*, t), \quad \chi_{[\nu_\epsilon,1]}(t) \int_0^1 \partial_\gamma q_0(\xi_s, t)\, ds \to \partial_\gamma q_0(\gamma_*, t),$$

pointwise a.e. on $t \in (0,1)$, and they are also majored as

$$\left|\chi_{[\nu_\epsilon,1]}(t) \int_0^1 \partial_\gamma \ell_0(\xi_s, t)\, ds\right| \le h_3(t), \quad \left|\chi_{[\nu_\epsilon,1]}(t) \int_0^1 \partial_\gamma q_0(\xi_s, t)\, ds\right| \le h_4(t).$$

Since $(1-t)^{-1/2}, |\log t|(1-t)^{-1/2} \in L^1_t(0,1)$, it follows that $h_3, h_4 \in L^1_t(0,1)$. By Lebesgue dominated convergence theorem on $t \in (0,1)$, rearrange (4.84) as the quotients in $\gamma \in J$ and the limits can be computed as

$$\begin{aligned}
\lim_{\gamma\to\gamma_*} \int_{\nu_\epsilon}^1 \frac{\ell(0,\gamma,t) - \ell(0,\gamma_*,t)}{\gamma - \gamma_*}\, dt &= \lim_{\epsilon\to 0^+} \int_{\nu_\epsilon}^1 \int_0^1 \partial_\gamma \ell_0(\xi_s, t)\, ds\, dt \\
&= \lim_{\epsilon\to 0^+} \int_0^1 \left[\chi_{[\nu_\epsilon,1]}(t) \int_0^1 \partial_\gamma \ell_0(\xi_s, t)\, ds\right] dt \\
&= \int_0^1 \partial_\gamma \ell_0(\gamma_*, t)\, dt, \\
\lim_{\gamma\to\gamma_*} \int_{\nu_\epsilon}^1 \frac{q(0,\gamma,t) - q(0,\gamma_*,t)}{\gamma - \gamma_*}\, dt &= \lim_{\epsilon\to 0^+} \int_{\nu_\epsilon}^1 \int_0^1 \partial_\gamma q_0(\xi_s, t)\, ds\, dt \\
&= \lim_{\epsilon\to 0^+} \int_0^1 \left[\chi_{[\nu_\epsilon,1]}(t) \int_0^1 \partial_\gamma q_0(\xi_s, t)\, ds\right] dt \\
&= \int_0^1 \partial_\gamma q_0(\gamma_*, t)\, dt.
\end{aligned} \tag{4.86}$$

To finish, we synthesize all pieces. Since the period is fixed as $L > 0$, it has $T(\epsilon, \gamma) = T(0, \gamma_*) = L$. By the vanishing terms (4.60) and the limits (4.83) and (4.86), as $\epsilon \to 0^+$, take the limit of the period quotient (4.58) with integrand (4.64), and it follows that

$$\begin{aligned}
0 &= \lim_{\epsilon\to 0^+} \frac{T(\epsilon,\gamma) - T(0,\gamma_*)}{\epsilon} \\
&= \lim_{\epsilon\to 0^+} 2\int_{\nu_\epsilon}^1 \frac{\ell(\epsilon,\gamma,t) - \ell(0,\gamma_*,t)}{\epsilon}\, dt - \lim_{\epsilon\to 0^+} \frac{2}{\epsilon} \int_0^{\nu_\epsilon} \ell(0,\gamma_*,t)\, dt \\
&= 2\int_0^1 \partial_\epsilon \ell(0,\gamma_*,t)\, dt + 2\left(\lim_{\epsilon\to 0^+} \frac{\gamma - \gamma_*}{\epsilon}\right) \int_0^1 \partial_\gamma \ell_0(\gamma_*, t)\, dt
\end{aligned}$$

i.e.

$$\lim_{\epsilon\to 0^+} \frac{\gamma - \gamma_*}{\epsilon} = \frac{-\int_0^1 \partial_\epsilon \ell(0,\gamma_*,t)\, dt}{\int_0^1 \partial_\gamma \ell_0(\gamma_*, t)\, dt}. \tag{4.87}$$

Similarly, by (4.60), (4.83) and (4.86), take $\epsilon \to 0^+$ of the mass quotient (4.59) and use (4.87), so the formal right derivative of $Q(\epsilon)$ at $\epsilon = 0$ can be computed as

$$\begin{aligned}
\left.\frac{dQ}{d\epsilon}\right|_{\epsilon=0^+} &= \lim_{\epsilon\to 0^+} \frac{Q(\epsilon,\gamma) - Q(0,\gamma_*)}{\epsilon} \\
&= -\lim_{\epsilon\to 0^+} 2\int_{\nu_\epsilon}^{1} \frac{q(\epsilon,\gamma,t) - q(0,\gamma_*,t)}{\epsilon}\,dt + \lim_{\epsilon\to 0^+} \frac{2}{\epsilon}\int_0^{\nu_\epsilon} q(0,\gamma_*,t)\,dt \\
&= -2\int_0^1 \partial_\epsilon q(0,\gamma_*,t)\,dt - 2\left(\lim_{\epsilon\to 0^+}\frac{\gamma-\gamma_*}{\epsilon}\right)\int_0^1 \partial_\gamma q_0(\gamma_*,t)\,dt \\
&= -2\int_0^1 \partial_\epsilon q(0,\gamma_*,t)\,dt + 2\frac{\int_0^1 \partial_\gamma q_0(\gamma_*,t)\,dt}{\int_0^1 \partial_\gamma \ell_0(\gamma_*,t)\,dt}\int_0^1 \partial_\epsilon \ell(0,\gamma_*,t)\,dt.
\end{aligned} \tag{4.88}$$

Next, we relate the integral quotient in (4.88) to derivative formula. Fix $\gamma_* \in J$, for every small $k \in \mathbb{R}$ such that $\gamma = \gamma_* + k \in J$, use (4.56) to write the difference quotients at $\epsilon = 0$ and apply fundamental theorem of calculus to a parameter $s \in [0,1]$, such that

$$\begin{aligned}
\frac{T(0,\gamma_*+k) - T(0,\gamma_*)}{k} &= \int_0^1 \frac{\ell(0,\gamma_*+k,t) - \ell(0,\gamma_*,t)}{k}\,dt \\
&= \int_0^1\int_0^1 \partial_\gamma \ell_0(\gamma_*+sk,t)\,ds\,dt, \\
\frac{Q(0,\gamma_*+k) - Q(0,\gamma_*)}{k} &= -\int_0^1 \frac{q(0,\gamma_*+k,t) - q(0,\gamma_*,t)}{k}\,dt \\
&= -\int_0^1\int_0^1 \partial_\gamma q_0(\gamma_*+sk,t)\,ds\,dt.
\end{aligned} \tag{4.89}$$

Since the integrands are majorized by (4.85) and $h_3, h_4 \in L^1_s(0,1) \cap L^1_t(0,1)$, use limit (4.86) and apply dominated convergence theorem on $t \in (0,1)$, so the limits of (4.89) take the form

$$\begin{aligned}
\lim_{k\to 0} \frac{T(0,\gamma_*+k) - T(0,\gamma_*)}{k} &= \int_0^1 \partial_\gamma \ell_0(\gamma_*,t)\,dt, \\
\lim_{k\to 0} \frac{Q(0,\gamma_*+k) - Q(0,\gamma_*)}{k} &= -\int_0^1 \partial_\gamma q_0(\gamma_*,t)\,dt.
\end{aligned} \tag{4.90}$$

Equivalently, since $\gamma_* = \gamma_*(L)$ at $\epsilon = 0$, let $h \in \mathbb{R}$ be small such that $\gamma = \gamma_*(L+h) \in J$. Write limiting period $T_*(L) := T(0,\gamma_*(L))$ and limiting mass $Q_*(L) := Q(0,\gamma_*(L))$ as one-variable functions of $L > 0$. Thus, the limits (4.90) can also be written as

$$\begin{aligned}
\lim_{h\to 0} \frac{T_*(L+h) - T_*(L)}{\gamma_*(L+h) - \gamma_*(L)} &= \lim_{h\to 0} \frac{T(0,\gamma_*(L+h)) - T(0,\gamma_*(L))}{\gamma_*(L+h) - \gamma_*(L)} = \int_0^1 \partial_\gamma \ell_0(\gamma_*,t)\,dt, \\
\lim_{h\to 0} \frac{Q_*(L+h) - Q_*(L)}{\gamma_*(L+h) - \gamma_*(L)} &= \lim_{h\to 0} \frac{Q(0,\gamma_*(L+h)) - Q(0,\gamma_*(L))}{\gamma_*(L+h) - \gamma_*(L)} = -\int_0^1 \partial_\gamma q_0(\gamma_*,t)\,dt.
\end{aligned}$$

Since the period is fixed, it has $h = (L+h) - L = T_*(L+h) - T_*(L)$. Thus, the derivative $Q_*'(L)$ can be computed using definition, such that

$$\frac{dQ_*}{dL} = \lim_{h\to 0} \frac{Q_*(L+h) - Q_*(L)}{h} = \frac{\displaystyle\lim_{h\to 0} \frac{Q_*(L+h) - Q_*(L)}{\gamma_*(L+h) - \gamma_*(L)}}{\displaystyle\lim_{h\to 0} \frac{T_*(L+h) - T_*(L)}{\gamma_*(L+h) - \gamma_*(L)}} = \frac{-\int_0^1 \partial_\gamma q_0(\gamma_*, t)\, dt}{\int_0^1 \partial_\gamma \ell_0(\gamma_*, t)\, dt}. \tag{4.91}$$

Therefore, the integral quotient in (4.88) can be replaced by the exact derivative using (4.91), and the right derivative at $\epsilon = 0$ is computed as

$$\left.\frac{dQ}{d\epsilon}\right|_{\epsilon=0^+} = -2\int_0^1 \partial_\epsilon q(0, \gamma_*, t)\, dt - 2\frac{dQ_*}{dL}\int_0^1 \partial_\epsilon \ell(0, \gamma_*, t)\, dt.$$

This completes the proof of this lemma. □

Next, we can prove the monotonicity. Different from the even case, the use of exact limiting mass (5.10) is valid but not necessary since the expressions end up being some sign-definite integrals with different power in logarithm function, so we can directly use square-completion trick as below.

Proposition 4.6. *For fixed $L > 0$, the odd wave mapping $\omega \mapsto Q$ is monotonically decreasing as $\omega \to 1^-$.*

Proof. Let $\epsilon = 1 - \omega \geq 0$ and fix $L > 0$ on the odd branch $T(\epsilon, \gamma(\epsilon)) = L$. We define sign-definite integrals as

$$\begin{aligned}
N_0 &:= \int_0^1 \frac{1}{\sqrt{1-t}(\gamma_* - t)^{3/2}}\, dt > 0,\\
N_1 &:= \int_0^1 \frac{\log t}{\sqrt{1-t}(\gamma_* - t)^{3/2}}\, dt < 0,\\
N_2 &:= \int_0^1 \frac{(\log t)^2}{\sqrt{1-t}(\gamma_* - t)^{3/2}}\, dt > 0.
\end{aligned}$$

By lemma 4.5, and the right derivative of Q can be computed as

$$\left.\frac{dQ}{d\epsilon}\right|_{\epsilon=0^+} = -2\int_0^1 \partial_\epsilon q(0, \gamma_*, t)\, dt - 2\frac{dQ_*}{dL}\int_0^1 \partial_\epsilon \ell(0, \gamma_*, t)\, dt. \tag{4.92}$$

where integrands are computed in (4.66) and evaluated at $(0, \gamma_*, t)$, so the integral can be represented as

$$\int_0^1 \partial_\epsilon \ell(0, \gamma_*, t)\, dt = -\frac{1}{2}N_1, \quad \int_0^1 \partial_\epsilon q(0, \gamma_*, t)\, dt = -\frac{1}{2}N_2.$$

By (4.91) and (4.67), the derivative of limiting mass can be represented as

$$\frac{dQ_*}{dL} = \frac{-\int_0^1 \partial_\gamma q_0(\gamma_*, t)\, dt}{\int_0^1 \partial_\gamma \ell_0(\gamma_*, t)\, dt} = -\frac{N_1}{N_0}$$

Thus, compute the derivative (4.92) by completing the square, such that

$$\begin{aligned}
\left.\frac{dQ}{d\epsilon}\right|_{\epsilon=0^+} = N_2 - \frac{N_1^2}{N_0} &= N_2 - 2\left(\frac{N_1}{N_0}\right) N_1 + \left(\frac{N_1}{N_0}\right)^2 N_0 \\
&= \int_0^1 \left(\log t - \frac{N_1}{N_0}\right)^2 \frac{1}{\sqrt{1-t}(\gamma_* - t)^{3/2}}\, dt \\
&= \int_0^1 \frac{[\log t + Q_*'(L)]^2}{\sqrt{1-t}(\gamma_* - t)^{3/2}}\, dt
\end{aligned}$$

i.e.

$$\left.\frac{dQ}{d\omega}\right|_{\omega=1^-} = -\left.\frac{d}{d\epsilon} Q(\epsilon, \gamma(\epsilon))\right|_{\epsilon=0^+} < 0. \tag{4.93}$$

as desired. □

5 Numerical Methods

5.1 Main Numerical Results

By using accurate numerical approximations based on the first invariant (1.7) and the period function (1.11), we can compute solutions of the implicit equation $T(\mathcal{E}_L(\omega), \omega) = L$ for a fixed spatial period $L > 0$ and the approximations of the spatial profile ϕ of the perodic wave.

Figure 5.1 shows the corresponding results for the even wave satisfying (1.9) with $x_0 = 0$. The left panel plots $\tilde{\mathcal{E}}_L := \mathcal{E}_L(\omega) - \mathcal{E}_\omega$ versus ω in $(\omega_L, 1)$ for $L = 2\pi, 3\pi, 4\pi$ and the right panel shows the spatial profile $\phi = \phi(x)$ versus x for $L = 4\pi$ and $\omega = 0.3, 0.6, 0.9$. Numerical inaccuracies in the computations occur near $\omega = 1$ and the end points in the numerical data on the left panel are shown by solid dots. The spatial profile of the even periodic wave becomes peaked as $\omega \to 1$. Solving (1.5) for $\omega = 1$ yields the peaked profile

$$\omega = 1: \quad \phi(x) = \frac{\cosh\left(\frac{L}{2} - |x|\right)}{\cosh\left(\frac{L}{2}\right)}, \qquad x \in \left[-\frac{L}{2}, \frac{L}{2}\right], \tag{5.1}$$

which is shown on the right panel by dashed line. The corresponding energy level can be computed as

$$\tilde{\mathcal{E}}_L(\omega = 1) = -\frac{1}{2\cosh^2\left(\frac{L}{2}\right)}, \tag{5.2}$$

which is shown on the left panel by open dots. An interpolation between the right solid dot and the open dot for (5.2) is shown by dashed line.

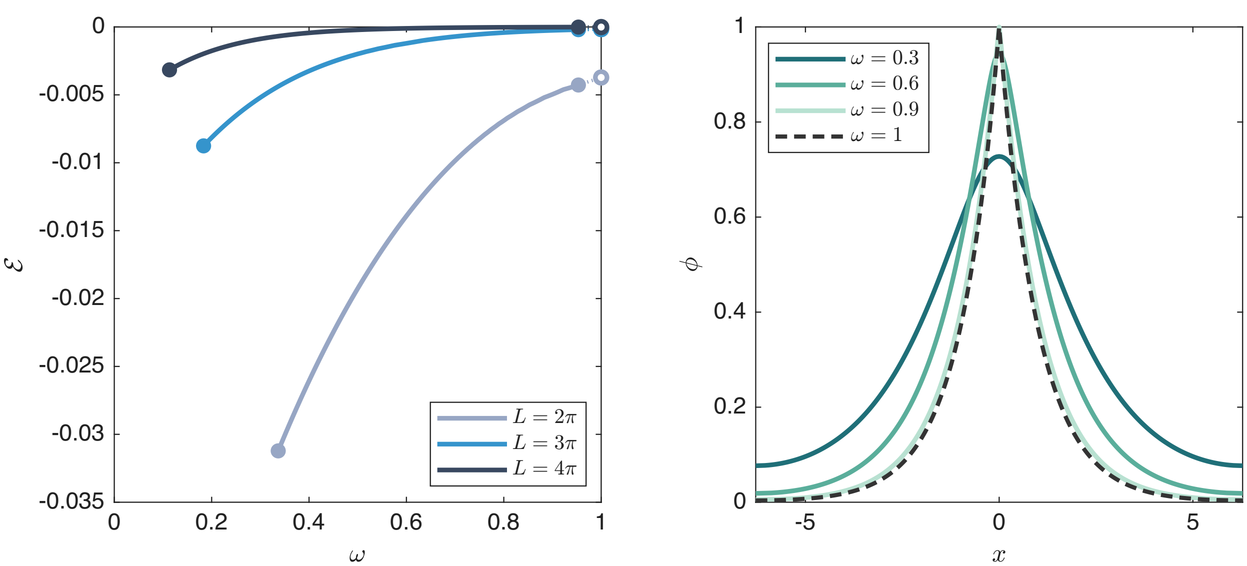

Figure 5.1: Numerical approximations for the even waves satisfying (1.9). Left: the dependence of $\tilde{\mathcal{E}}_L$ versus ω for $L = 2\pi, 3\pi, 4\pi$. Right: the spatial profile ϕ versus x for $\omega = 0.3, 0.6, 0.9$ and $L = 4\pi$.

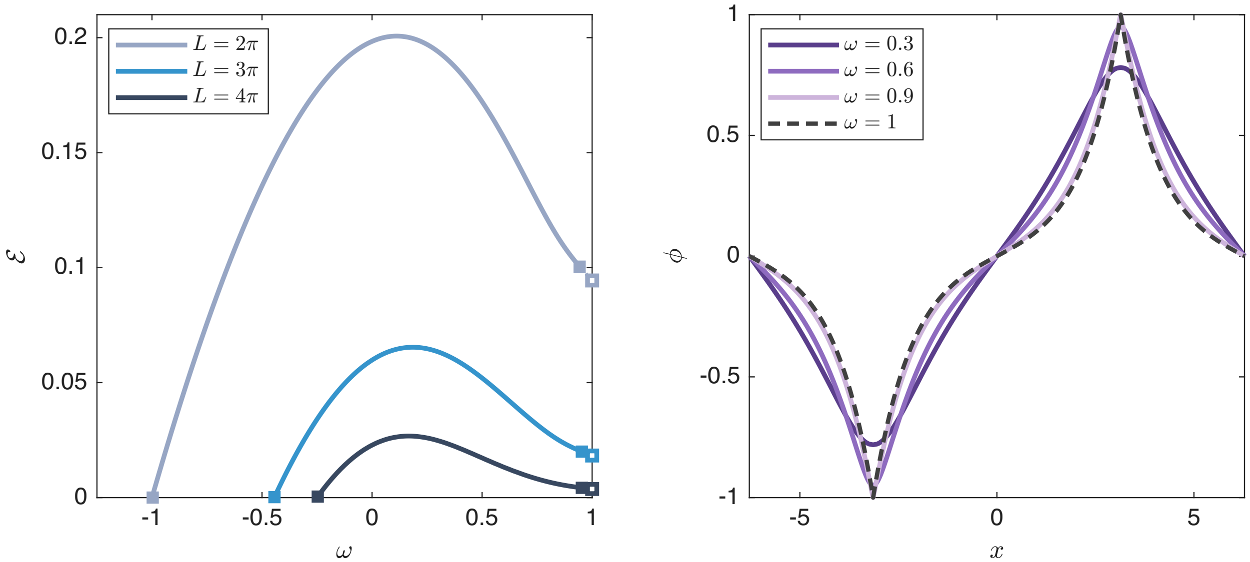

Figure 5.2: Numerical approximations for the odd waves satisfying (1.10). Left: the dependence of $\tilde{\mathcal{E}}_L$ versus ω for $L = 2\pi, 3\pi, 4\pi$. Right: the spatial profile ϕ versus x for $\omega = 0.3, 0.6, 0.9$ and $L = 4\pi$.

Figure 5.2 shows the corresponding results for the odd wave satisfying (1.10) with $x_0 = 0$ for $\omega \in (\Omega_L, 1)$ with $\Omega_L < 0$. We note the non-monotone dependence of $\tilde{\mathcal{E}}_L := \mathcal{E}_L(\omega) - \mathcal{E}_\omega$ versus ω on the left panel, which is not an obstacle to our analysis. The spatial profile of the periodic wave becomes peaked as $\omega \to 1$. Solving (1.5) for $\omega = 1$ yields the odd spatial profile in the form:

$$\omega = 1: \quad \phi(x) = \begin{cases} -\dfrac{\sinh\left(x + \frac{L}{2}\right)}{\sinh\left(\frac{L}{4}\right)} & x \in \left[-\frac{L}{2}, -\frac{L}{4}\right] \\ \dfrac{\sinh x}{\sinh\left(\frac{L}{4}\right)} & x \in \left[-\frac{L}{4}, \frac{L}{4}\right] \\ \dfrac{\sinh\left(\frac{L}{2} - x\right)}{\sinh\left(\frac{L}{4}\right)} & x \in \left[\frac{L}{4}, \frac{L}{2}\right] \end{cases}, \tag{5.3}$$

which is shown on the right panel by dashed line. The corresponding energy level can be computed as

$$\tilde{\mathcal{E}}_L(\omega = 1) = \frac{1}{2\sinh^2\left(\frac{L}{4}\right)}, \tag{5.4}$$

which is shown on the left panel by open dots. The end points in the numerical data on the left panel are shown by solid dots. An interpolation between the right solid dot and the open dot for (5.4) is shown by dotted line.

By using the numerical approximation of the spatial profile ϕ, we can also compute the mass $Q(\phi)$ for a fixed spatial period $L > 0$ and plot it versus ω to verify the sharp criterion for the energetic stability of the periodic waves given by Theorem 1.4. Figure 5.3 plots $Q(\phi)$

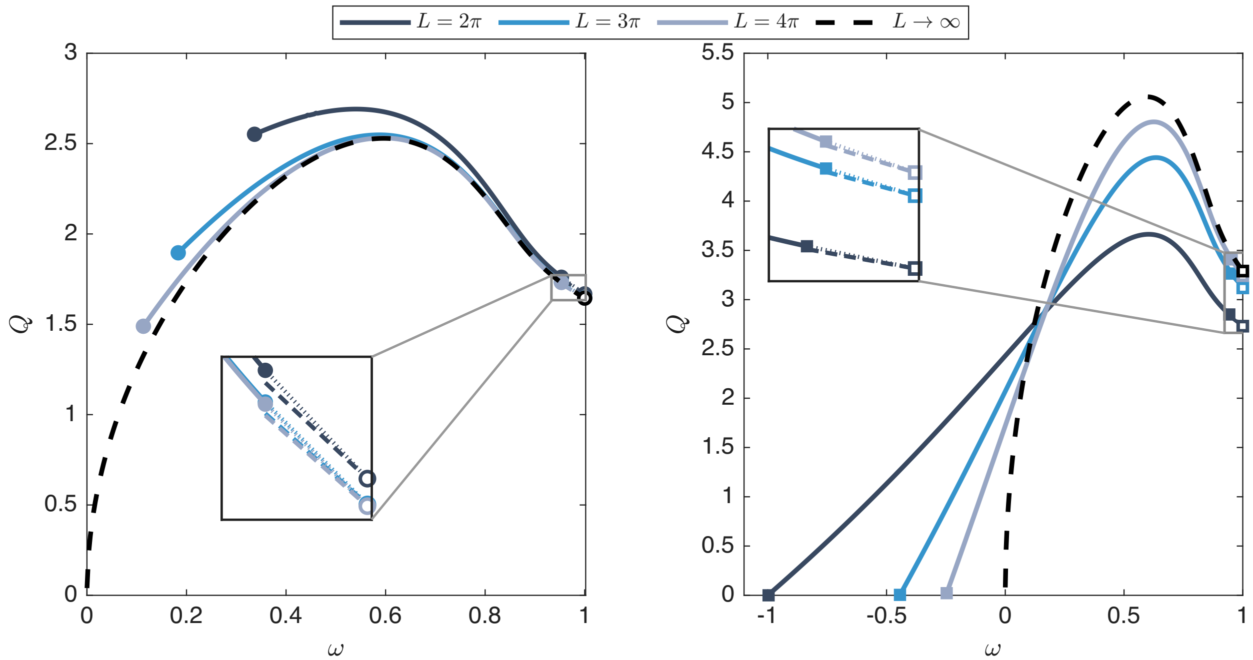

Figure 5.3: Dependence of $Q(\phi)$ versus ω for $L = 2\pi, 3\pi, 4\pi$ and in the limit $L \to \infty$ (dashed lines). Left panel: the even wave satisfying (1.9). Right panel: the odd wave satisfying (1.10). The magnified sub-panels use the coloured dashed lines to show the exact tangent lines computed using left derivative formulas 4.3 and 4.50, and the difference of slope comparing to the interpolated secant lines is $\sim 10^{-1}$.

versus ω for $L = 2\pi, 3\pi, 4\pi$. The dashed line shows the dependence of $Q(\phi)$ in the limit $L \to \infty$, which corresponds to the solitary waves. The left panel presents the mapping $\omega \to Q(\phi)$ for the even wave satisfying (1.9) and the right panel presents the same for the odd wave satisfying (1.10). The numerical inaccuracies occur near $\omega = 1$ and the end points of the numerical data are shown by solid dots. By using (5.1) and (5.3), we are able to compute $Q(\phi)$ analytically at $\omega = 1$ for the peaked waves, see (5.8) and (5.10) below, and show the result in Figure 5.3 by open dots. An interpolation between the right solid dot and the open dot is shown by dotted line.

Remark 5.1. *The numerical data in Figures 1.2, 5.1, 5.2, and 5.3 are obtained with high numerical accuracy, controlled within* 10^{-8} *error, since the numerical error only arises in the computation of the period function* $T(\mathcal{E}, \omega)$ *and the wave profile* $\phi(x)$ *from the corresponding integrals. The dependence of* $Q(\phi)$ *versus* ω *in the limit* $L \to \infty$ *to the solitary wave shown in Figure 5.3 contradicts the claim from [18, Figure 5] that the dependence is monotonically increasing near* $\omega = 0$ *and* $\omega = 1$ *and decreasing for* $\omega \in (\omega_1, \omega_2)$ *for some* $0 < \omega_1 < \omega_2 < 1$*. Although the numerical data on the dependence of* $Q(\phi)$ *on* ω *was consistent with the numerical approximations of unstable eigenvalues in the spectral stability problem, see same Figure 5 in [18], we have found that the claim of stability of bright solitons near* $\omega = 1$ *in [18] is a numerical artefact. It is related with the center-difference approximations of the second-*

order derivatives with a large stepsize, which were used in [18]. By reducing the stepsize or performing computations with adaptive methods directly from (1.7) and (1.11), we have found that $Q(\phi)$ is monotonically decreasing in ω near $\omega = 1$. Although our numerical data has a tiny gap near $\omega = 1$ due to the lack of numerical accuracy, comparison between the last numerical data (solid dots) and the analytically computed limiting value of $Q(\phi)$ at $\omega = 1$ (open dots) suggest the monotone decrease of $Q(\phi)$ near $\omega = 1$.

5.2 Technical Highlights

Given a fixed $\omega \in (0,1)$, the energy level of the homoclinic orbit $\mathcal{E}_\omega \in (0,1)$ is computed, and then the period function $T(\mathcal{E},\omega)$ for the even and odd periodic waves is approximated separately as in (2.17) and (2.19), respectively. The period function is plotted on Figure 1.2.

For the even waves, since the slope of the period function diverges as $\mathcal{E} \to \mathcal{E}_\omega^-$, the grid on $(0,\mathcal{E}_\omega)$ is defined in two regions $(\mathcal{E}_\omega - 10^{-3}, \mathcal{E}_\omega)$ with 2000 equally spaced grid points and $(0, \mathcal{E}_\omega - 10^{-3})$ with 300 equally spaced grid points. For the odd waves, the grids are defined analogously on $(\mathcal{E}_\omega, \mathcal{E}_\omega + 10^{-2})$ with 100 grid points and $(\mathcal{E}_\omega + 10^{-2}, 0.5)$ with 300 grid points. We evaluate the integrals with absolute and relative tolerances $\epsilon_{\text{abs}} = 10^{-10}$ and $\epsilon_{\text{rel}} = 10^{-8}$ respectively. Selected values $\omega = 0.3, 0.5, 0.7, 0.9$ are plotted in Figure 1.2 with $T = 2\pi\sqrt{\frac{1-\omega}{2\omega}}$ at $\mathcal{E} = 0$ represented by solid dots.

Once the period function $T(\mathcal{E},\omega)$ is computed, we fix the spatial period $L > 0$ and determine the uniquely defined energy level $\mathcal{E}_L(\omega)$ from the root of $T(\mathcal{E}_L(\omega),\omega) = L$. This is possible due to the monotonicity of the period function with respect to $\mathcal{E}$ in Theorem 1.3. We use Newton's root-finding method for a grid $\{\omega_j\}_{j=1}^M$of values of ω in either $(\omega_L, 1)$ or $(\Omega_L, 1)$, see Theorems 1.1 and 1.3. We thus obtain the values $\{\mathcal{E}_j\}_{j=1}^M$ for $\mathcal{E}_L(\omega_j)$, which are plotted on the left panels of Figure 1.3 and Figure 1.4 relative to $\mathcal{E}_\omega$, for $\tilde{\mathcal{E}}_L(\omega) = \mathcal{E}_L(\omega) - \mathcal{E}_\omega$. Thus, the solid dots for $\omega = \omega_L$ correspond to $\tilde{\mathcal{E}_L} = -\mathcal{E}_\omega$ in Figure 1.3 and the solid dots for $\omega = \Omega_L$ correspond to $\tilde{\mathcal{E}}_L(\omega) = 0$ on Figure 1.4.

For the computed set $\{(\mathcal{E}_i, \omega_i)\}_{i=1}^M$, the profile $\phi = \phi(x)$ of the even periodic wave satisfying (1.9) with $x_0 = 0$ is obtained by numerical integration of

$$x = F_{\text{even}}(\phi) = \int_\phi^M \frac{d\varphi}{\sqrt{2\mathcal{E}_L(\omega) - (\omega - \varphi^2) - (1-\omega)\log\frac{1-\omega}{1-\varphi^2}}}, \qquad \phi \in [m, M], \tag{5.5}$$

where m and M is obtained from two positive roots of $V(\phi) = \mathcal{E}_L(\omega)$ for $\omega \in (\omega_L, 1)$ and $\mathcal{E}_L \in (0, \mathcal{E}_\omega)$. The solution profile is defined implicitly as $x = F_{\text{even}} \in [0, \frac{L}{2}]$ with $\phi(0) = M$ and $\phi(\frac{L}{2}) = m$. It is extended from $[0, \frac{L}{2}]$ to $[-\frac{L}{2}, 0]$ by using the even reflection: $\phi(-x) = \phi(x)$. This yields the wave profiles on the right panel of Figure 1.3. The dashed line shows the peaked profile at $\omega = 1$ given analytically by (5.1).

For the odd periodic wave, the profile $\phi = \phi(x)$ satisfying (1.10) is obtained by numerical integration of

$$x = F_{\text{odd}}(\phi) = \int_0^{\phi} \frac{d\varphi}{\sqrt{2E_L(\omega) - (\omega - \varphi^2) - (1-\omega)\log\left(\frac{1-\omega}{1-\varphi^2}\right)}}, \qquad \phi \in [0, M]. \tag{5.6}$$

where M is obtained from the only positive root of $V(\phi) = \mathcal{E}_L(\omega)$ for $\omega \in (\Omega_L, 1)$ and $\mathcal{E}_L(\omega) \in (\mathcal{E}_\omega, \infty)$. The solution profile is defined implicitly as $x = F_{\text{odd}}(\phi) \in [0, \frac{L}{4}]$ with $\phi(0) = 0$ and $\phi(\frac{L}{4}) = M$. It is extended from $[0, \frac{L}{4}]$ by using the symmetries of the odd periodic wave $\phi(-x) = -\phi(x) = -\phi(\frac{L}{2} - x)$. This yields the wave profiles on the right panel of Figure 1.4. The dashed line shows the peaked profile at $\omega = 1$ given analytically by (5.3).

We compute the mass $Q(\phi)$ shown in Figure 5.3 versus ω by using the integration in the ϕ variable. For the even periodic wave, we use

$$Q(\phi) = -2\int_0^{L/2} \log(1-\phi^2)\, dx = 2\int_m^M \frac{\log(1-\phi^2)}{\sqrt{2E_L(\omega) - (\omega - \phi^2) - (1-\omega)\log\left(\frac{1-\omega}{1-\phi^2}\right)}}\, d\phi. \tag{5.7}$$

Computing the integral numerically for $\{(\mathcal{E}_i, \omega_i)\}_{i=1}^M$ yields the left panel of Figure 5.3. The numerical data are again missing near $\omega = 1$ and the last available data is shown by the solid dots, for which the accuracy of 10^{-8} is guaranteed. The open dots show the limiting values of $Q(\phi)$ at $\omega = 1$, which can be computed analytically as

$$\omega = 1: \quad Q(\phi) = 2L\log\left[2\cosh\left(\frac{L}{2}\right)\right] - L^2 + \frac{\pi^2}{6} - \mathrm{Li}_2(e^{-2L}), \tag{5.8}$$

where Li_2 denotes the dilogarithm function

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\ln(1-u)}{u}\, du.$$

Interpolation between the last available data (right solid dots) and the limiting value of $Q(\phi)$ at $\omega = 1$ (open dots) is shown by the dotted line on Figure 5.3.

The dashed line on the left panel of Figure 5.3 shows the limiting value of $Q(\phi)$ versus ω in the soliton case with $L = \infty$, for which the integral for $Q(\phi)$ is still computed on the compact interval. The dependence of $Q(\phi)$ versus ω is similar to the periodic case $L < \infty$ and displays a single maximum before the peak for which

$$\omega = 1, \quad L = \infty: \quad Q(\phi) = \frac{\pi^2}{6}.$$

For the odd periodic wave, we use

$$Q(\phi) = -4\int_0^{L/4} \log(1-\phi^2)\, dx = 4\int_0^M \frac{\log(1-\phi^2)}{\sqrt{2\mathcal{E}_L(\omega) - (\omega - \phi^2) - (1-\omega)\log\frac{1-\omega}{1-\phi^2}}}\, d\phi. \tag{5.9}$$

Computing the integral numerically for $\{(\mathcal{E}_i, \omega_i)\}_{i=1}^{M}$ yields the right panel of Figure 5.3. The limiting value of $Q(\phi)$ at $\omega = 1$ is computed analytically as

$$\omega = 1: \quad Q(\phi) = 2L \log\left[2\sinh\left(\frac{L}{4}\right)\right] - \frac{L^2}{2} + \frac{\pi^2}{3} - 2\mathrm{Li}_2(e^{-L}), \tag{5.10}$$

We note that

$$\omega = 1, \quad L = \infty: \quad Q(\phi) = \frac{\pi^2}{3}.$$

which is double compared to the case of the even periodic wave. This corresponds to the fact that the odd periodic wave represents two solitons on a single period for large L.

L	$Q_{\text{even}}(\omega = 1)$	$Q_{\text{odd}}(\omega = 1)$
2π	1.66837567259328	2.73100651970082
3π	1.64645514903036	3.11961052401896
4π	1.64502171315626	3.24288332619890

Table 1: The numerical values of $Q(\phi)$ used in Figure 5.3 for $\omega = 1$.

Table 1 represents the numerical values of $Q(\phi)$ used in Figure 5.3 for $\omega = 1$. These numerical values are computed from (5.8) and (5.10).

Finally, we expand Remark 5.1 to discuss the three-branched behaviour of $Q(\phi)$ versus ω in the soliton limit $L = \infty$ observed in [18] and disputed in Figure 5.3. We cannot reproduce three-branched behaviour by using (5.8) and (5.10). Even if we take fewer number of grid points, we would evaluate $Q(\phi)$ with a lower accuracy but still observe the two-branched behaviour of $Q(\phi)$ versus ω in Figure 5.3.

The reason for the three-branched behaviour of $Q(\phi)$ observed in [18] is due to the finite-difference approximation applied to the differential equation (1.5) and to the Hessian operator $\mathcal{L}$ in (1.12) with the uniform grid of x values. The larger grid spacing leads to inaccurate computations of ϕ near the maximum $\phi(0) = M$ and results in highly inaccurate computations of $Q(\phi)$.

We fix $\omega \in (0, 1)$ and consider the differential equation (1.5) on the truncated interval $[-L, L]$ with $L = 20$. Since the bright solitons decay exponentially to zero at infinity, we can use the Dirichlet boundary conditions $\phi(\pm L) = 0$. We replace $[-L, L]$ by the uniform grid of N points $\{x_i\}_{i=1}^{N}$ with the spacing $\Delta x = \frac{2L}{N-1}$ and compute approximations for the solution profile $\{\phi\}_{i=1}^{N}$ with $\phi_1 = \phi_N = 0$. The second derivative can be constructed using the central difference method as $\{(D^2\phi)_j\}_{j=2}^{N-1}$ given by

$$(D^2\phi)_j = \frac{\phi_{j-1} - 2\phi_j + \phi_{j+1}}{(\Delta x)^2}, \qquad j = 2, \dots, N-1$$

The residual of the differential equation (1.5) is defined by

$$R_j = (1 - \phi_j^2)(D^2\phi)_j - (\omega - \phi_j^2)\phi_j, \qquad j = 2, \dots, N-1,$$

and we introduce the mapping $T : \mathbb{R}^{N-2} \to \mathbb{R}^{N-2}$ such that $T(\phi) = R$. The first derivative of the mapping is given by the Jacobian matrix $J = \Delta T \in \mathbb{R}^{N-2\times N-2}$ with the nonzero elements given by

$$J_{j,j+1} = \frac{1-\phi_j^2}{(\Delta x)^2}, \qquad J_{j,j} = -\frac{2(1-\phi_j^2)}{(\Delta x)^2} - 2\phi_j (D^2\phi)_j - \omega + 3\phi_j^2, \quad 2 \le j \le N-2$$

To minimize the residual $\Phi(\phi) = \frac{1}{2}\|T(\phi)\|^2$, we implement the linear Newton's method in the iterations $\{\phi^{(k)}\}_{k=0}^{\infty}$ defined by $J\big(\phi^{(k+1)} - \phi^{(k)}\big) = -T(\phi^{(k)})$ starting with a suitable initial guess

$$\phi^{(0)} = \min\{0.9, \sqrt{2\omega}\}\mathrm{sech}\,(\sqrt{\omega}x_j), \qquad j = 2, \dots, N-1$$

To avoid overshoot, we perform backtrack line search by starting from $a = 1$ and reducing to find $a \in (0, 1]$ that satisfies the decreasing condition

$$\Phi(\phi^{(k)} + a\psi^{(k)}) \le \Phi(\phi^{(k)})(1 - ca), \quad \text{where } \psi^{(k)} = -J^{-1}T(\phi^{(k)})$$

for a small $c = 10^{-4}$. When this is achieved, we accept and update the next iteration as $\phi^{(k+1)} = \phi^{(k)} + a\psi^{(k)}$, after which we compute $J^{k+1}, T(\phi^{(k+1)})$, and $\psi^{(k+1)}$. The algorithm is terminated when the convergence condition $\|T(\phi^{(k+1)})\|/\sqrt{N-2} \le \epsilon_{\rm tol}$ with a small $\epsilon_{\rm tol} = 10^{-8}$. This iterative method yields the solution profile $\{(x_j, \phi_j)\}_{j=1}^{N}$, from which we compute the mass integral $Q(\phi)$ by using the trapezoidal method.

Figure 5.4 shows the plot of $Q(\phi)$ versus ω for two spacings $\Delta x = 0.1$ and $\Delta x = 0.2$, compared to the dependence computed from (5.7) in the limit $L \to \infty$ (dashed line). The latter dependence is interpreted as the limit $\Delta x \to 0$ in the finite-difference method. The finite-difference approximation with $\Delta x > 0$ for the differential equation (1.5) leads to the three-branched behaviour reported in [18]. We computed the mass integral for the values of ω in $[0.005, 0.93]$ on an equally spaced grid of 100 points. Since the numerical data are not accurate near $\omega = 1$, we perform the quadratic extrapolation to extend the values of the mass integral from the last numerical data at $\omega = 0.93$ into the interval $[0.93, 1]$.

Thus, we conclude that the three-branched behaviour of $Q(\phi)$ versus ω is a numerical artefact of the finite-difference method.

References

[1] J. Albert and H. Arbunich, "Stability of bound states for regularized nonlinear Schrödinger equations, Stud. Appl. Math. **153** (2024) e12780 (37 pages).

[2] P. Antonelli, J. Arbunich, and C. Sparber, "Regularizing nonlinear Schrödinger equations through partial off-axis variations", SIAM J. Math. Anal. **51** (2019) 110–130.

[3] B. Buffoni and J. Toland, Analytic theory of global bifurcation, Princeton University Press, Princeton, 2016.

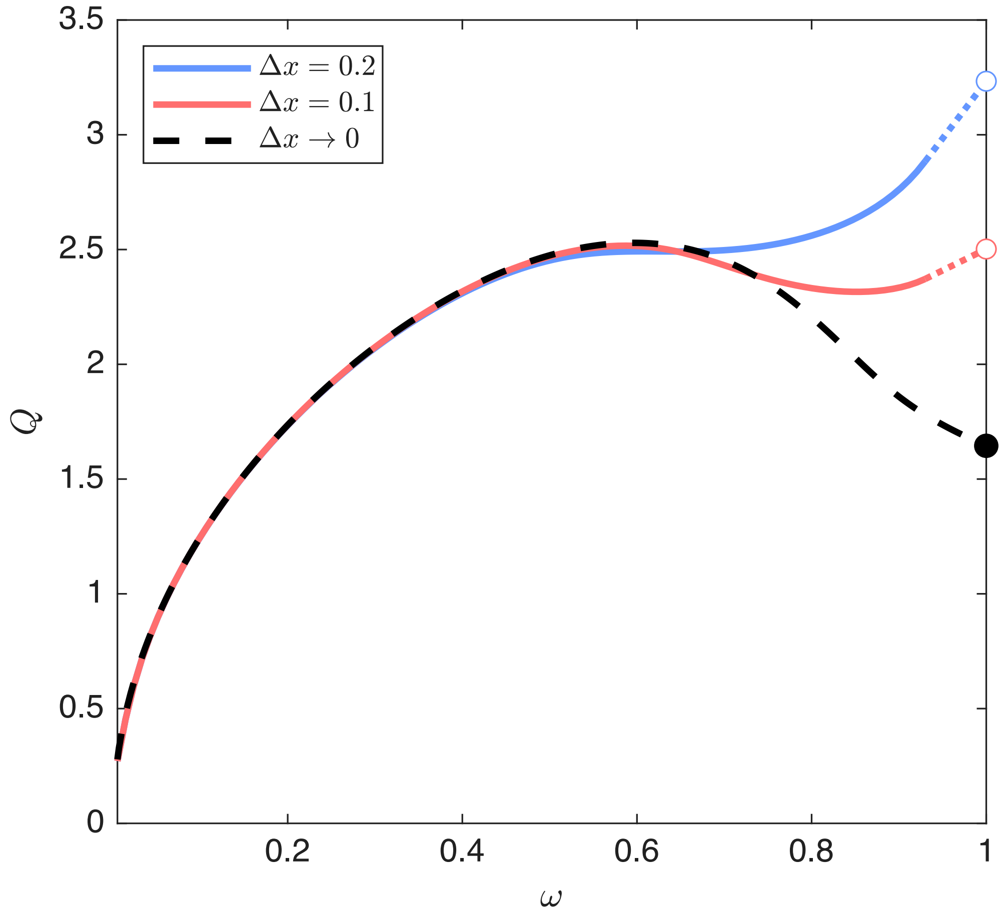

Figure 5.4: The dependence of the mass integral $Q(\phi)$ computed by the finite-difference method versus ω for $\Delta = 0.1, 0.2$. The dashed line shows the same dependence computed by using (5.7) for $L \to \infty$

[4] J.-H. Chang, C.-Y. Lin, R.-K. Lee, "Quantum harmonic oscillators with nonlinear effective masses in the weak density approximation", Phys. Scripta **97** (2022) 025205.

[5] C. Chicone, The monotonicity of the period function for planar Hamiltonian vector fields, J. Diff. Equat. **69** (1987), 310–321.

[6] M.S.P. Eastham, "The Spectral Theory of Periodic Differential Equations", Texts in Mathematics, Scottish Academic Press, Edinburgh; Hafner Press, New York, 1973.

[7] R. Feola and F. Iandoli, "Local well-posedness for quasi-linear NLS with large Cauchy data on the circle", Ann. I.H.Poincaré **36** (2019) 119–164.

[8] R. Feola and F. Iandoli, "Local well-posedness for the quasi-linear Hamiltonian Schrödinger equation on tori", J. Math. Pures Appl. **157** (2022) 243–281.

[9] T. Feola, B. Grébert, and F. Iandoli, "Long time solutions for quasilinear Hamiltonian perturbations of Schrödinger and Klein-Gordon equations on tori", Anal. PDE **16** (2023) 1133–1203.

[10] G. Fibich, "The Nonlinear Schrödinger Equation: Singular Solutions and Optical Collapse" (New York: Springer, 2015)

[11] T. Gallay and M. Hărăguş, "Orbital stability of periodic waves for the nonlinear Schrödinger equation", J. Dynam. Diff. Eqs. **19** (2007) 825–865.

[12] A Geyer and D. E. Pelinovsky, "Stability of Nonlinear Waves in Hamiltonian Dynamical Systems", Mathematical Surveys and Monographs **288** (AMS, Providence, 2025)

[13] A.D. Greentree, D. Richards, J.A. Vaccaro, A.V. Durant, S.R. de Echaniz, D.M. Segal, and J.P. Marangos, "Intensity-dependent dispersion under conditions of electromagnetically induced transparency in coherently prepared multistate atoms", Phys. Rev. A **67** (2003), 023818.

[14] M. Ifrim and D. Tataru, "Global solutions for 1d cubic dispersive equations, part III: The quasilinear Schrödinger flow", Inventiones Math. **242** (2025) 221–304

[15] C. E. Kenig, G. Ponce, and L. Vega, "The Cauchy problem for quasi-linear Schrödinger equations", Invent. Math. **158** (2004) 343–388.

[16] T. Kato, Perturbation theory for linear operators, Springer, Berlin, (1976).

[17] P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González, "The defocusing nonlinear Schrödinger equation: From dark solitons to vortices and vortex rings', (Society for Industrial and Applied Mathematics, Philadelphia, PA, 2015).

[18] P.G. Kevrekidis, D.E. Pelinovsky, and R.M. Ross, "Stability of smooth solitary waves under the intensity–dependent dispersion", IMA J. Appl. Math. **89** (2024) 989–1005.

[19] A.A. Koser, P.K. Sen, and P. Sen, "Effect of intensity dependent higher-order dispersion on femtosecond pulse propagation in quantum well waveguides", J. Mod. Opt. **56** (2009) 1812–1818.

[20] A. de Laire and S. Lopez-Martinez, "Existence and decay of traveling waves for the nonlocal Gross- Pitaevskii equation", Comm. PDEs **47** (2022) 1732–1794.

[21] A. de Laire and S. Lopez-Martinez, 'Gray and black solitons of nonlocal Gross-Pitaevskii equations: existence, monotonicity and nonlocal-to-local limit", Calc. Var. PDEs **64** (2025) 138 (45 pages).

[22] A. de Laire and E. Le Quiniou, "Exotic traveling waves for a quasilinear Schr¨odinger equation with nonzero background", arXiv: 2311.08918, version 2 (2025).

[23] C.Y. Lin, J.H. Chang, G. Kurizki, and R.K. Lee, "Solitons supported by intensity-dependent dispersion", Optics Letters **45** (2020), 1471–1474.

[24] W. Magnus and S. Winkler, Hill's equation, Tracts in Pure and Applied Mathematics, Wiley, New York, (1966).

[25] J. L. Marzuola, J. Metcalfe, and D. Tataru, "Quasilinear Schrödinger equations, II: small data and cubic nonlinearities", Kyoto J. Math. **54** (2014) 529–546.

[26] J. L. Marzuola, J. Metcalfe, and D. Tataru, "Quasilinear Schrödinger equations III: Large data and short time", Arch. Ration. Mech. Anal. **242** (2021) 1119–1175.

[27] A. Neves,"Floquet's theorem and the stability of periodic waves", J. Dyn. Diff. Equat., **21** (2009), 555-565.

[28] D. E. Pelinovsky and M. Plum, "Stability of black solitons in optical systems with intensity-dependent dispersion", SIAM J. Math. Anal. **56** (2024) 2521–2568

[29] D.E. Pelinovsky and M. Plum, "Dynamics of black solitons in a regularized nonlinear Schrödinger equation", Proceeding AMS **152** (2024) 1217–1231

[30] D. E. Pelinovsky, R. M. Ross, and P. G. Kevrekidis, "Solitary waves with intensity-dependent dispersion: variational characterization", J. Phys. A: Math. Theor. **54** (2021) 445701 (15 pages)

[31] M. Poppenberg, "Smooth solutions for a class of fully nonlinear Schrödinger type equations", Nonlinear Anal. **45** (2001) 723–741.

[32] R. M. Ross, P. G. Kevrekidis, and D. E. Pelinovsky, "Localization in optical systems with an intensity-dependent dispersion", Quart. Appl. Math. **79** (2021), 641–665.

[33] M. Salerno, "Quantum deformations of the discrete nonlinear Schrödinger equation", Phys. Rev. A **46** (1992) 4856-4859.

[34] F. Natali, D. E. Pelinovsky, S. Wang, "Stability of periodic waves in the model with intensity–dependent dispersion", arXiv:2603.28514 (2026).