

Stability of dark solitons in a bubble Bose-Einstein condensate

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The dynamic stability of dark solitons trapped on the surface of a two-dimensional spherical bubble is investigated. In this spherical geometry of the Bose-Einstein condensate, dark solitons are found to be unstable for the interaction parameter $\varepsilon \gtrsim 8.37$, since discrete angular modes drive snake instabilities, with the generation of vortex dipoles. We show analytically and numerically that, for each angular mode $m \geq 2$, there exists exactly one unstable mode whose dominance determines the number m of vortex dipoles. Time-dependent simulations confirm the formation of vortex dipoles.

Bose-Einstein condensed gases in spherical geometries have recently attracted attention due to experiments performed aboard the International Space Station in a microgravity environment with ultra-cold gases confined in spherical and/or ellipsoidal surfaces [1, 2]. These experimental investigations were designed based on previous theoretical work on shell-like potentials [3–5]. Ground-based experiments also revealed shell bubbles, by exploring two species [6], or intending to observe alternative two-dimensional (2D) closed geometries [7]. Such investigations have shed new light on the physics of low-dimensional quantum gases, especially in closed 2D shells. See Ref. [8] for a recent overview of the present status and perspectives on quantum gases in bubble traps. Several interesting problems have been studied in this context, considering fundamental physics properties in shell-like structures [9–14], vortex dynamics and stability [15–17], dipole interaction effects [18, 19], Berezinskii-Kosterlitz-Thouless (BKT) transition [20], thermodynamic properties of the gas adiabatic expansion from filled sphere to hollow one [21], and to our main interest, some attention has been paid to the properties of condensate mixtures trapped on a bubble [22–24]. Furthermore, the study of dimers on a spherical surface shows that the dimers are squeezed in the direction orthogonal to the center of mass motion, qualitatively changing its geometry, from 2D to one-dimensional (1D), leading to two-soliton motion on a bubble surface [25].

In the context of the properties of Bose-Einstein condensates (BECs) on spherical surfaces, a topic of particular interest concerns the existence, stability, and propagation of dark solitons [26]. Bubble BECs are realized when atoms are confined to a thin spherical shell, creating unique curvature effects that influence the dynamics of solitons. Dark solitons are shape-localized propagating waves manifested by depressions in the condensate density. On a bubble surface, geometric effects can alter the instability dynamics of dark solitons, as compared to their propagation on flat surfaces. The curved surface is expected to affect their dynamics, propagation speed, and stability. Unlike in flat quasi-1D or quasi-2D BECs, where dark solitons are susceptible to snake insta-

bility [27], decaying into vortices, the closed topology of a bubble is expected to alter the decay pathways, with the curvature introducing geometric constraints affecting the stability dynamics. Therefore, the soliton can experience strain due to the curvature, leading to contraction or expansion depending on the interactions.

The instability of a dark soliton stripe on a sphere follows a procedure similar to that in flat space. However, a sphere has no edges, such that the curvature influences perturbations that would be symmetric in a flat system. The most critical difference in a bubble refers to the fragmentation into vortices, as the resulting vortices are not free to move arbitrarily. The snake instabilities on a flat plane produce vortex-antivortex pairs that can move apart from each other and eventually are ejected from the high-density region. On a sphere, there is no edge to eject vortices as in flat space, with vortex-antivortex pairs remaining trapped on the surface, leading to complex dynamics like vortex-antivortex annihilation or formation of stable stationary patterns. The profound topological constraint implies that a single vortex is not allowed on a sphere, as vortices must exist in pairs of opposite circulation whose charges sum to zero.

Dark solitons in a flat BEC are known to oscillate and decay due to quantum fluctuations or dissipation. In quasi-1D traps, dark solitons can be metastable and decay due to transverse instabilities [28, 29]. Using harmonic traps, they were studied in [30, 31], both analytically and computationally, where it was found that the large-amplitude field modulations at a frequency resonant with the energy of a dark soliton give rise to a state with multiple vortices. The stability spectrum of the dark soliton contains complex frequencies, which disappear for sufficiently small numbers of atoms or for a large transverse confinement. The relationship between these complex modes and the snake instability was investigated numerically by real-time propagation, see also [32]. In ring geometries, dark solitons in BEC systems, introduced in [33], were further investigated in [34].

Two-component BECs were considered in [35], where the dark soliton exists in one of the condensate components and the soliton nodal plane is filled with the second

component. The filled solitons are stable for hundreds of milliseconds. By selectively removing the filling, one can make the soliton more susceptible to dynamical instabilities. For a condensate in a spherically symmetric potential, these instabilities cause the dark soliton to decay into stable vortex rings. By studying the oscillations and interactions of dark and dark-bright solitons, it was shown in [36] that the stability of solitons can be controlled in BECs, via confinement.

The aim of the research we present concerns a study on the spectral stability of dark solitons under discrete m -angular modes within a BEC confined on the surface of a bubble in the approximation of a 2D spherical hollow shell [15]. We show the occurrence of exactly one unstable mode for each $m \geq 2$. These unstable modes induce snake-like excitations, which break the condensate into pieces, as observed in [15]. As verified numerically, an m -dominated unstable mode is followed by the creation of m vortex-antivortex pairs, in agreement with our analytical predictions.

This letter is organized as follows. We first present the theoretical model for an atomic BEC confined to the surface of a perfect sphere, named a bubble. Dark solitons, together with the asymptotic approximations in the limit of small and large chemical potentials, are obtained analytically and numerically, supported by Supplemental Material (SM) [37]. We then report the analytical results of the corresponding stability analysis, demonstrating they are in excellent agreement with numerical observations. Finally, we report outcomes of the nonlinear dynamics of the system, from which it follows that the instabilities result in the formation of vortex dipoles on the dark soliton condensate.

Mathematical model — The present study is performed by assuming the condensate is trapped on the surface of a rigid spherical shell, aiming to mimic the cold-atom bubble experiments that are currently being performed in microgravity environments. In this approach, the system can be studied by considering a reduction of the 3D Gross-Pitaevskii equation (GPE) to a corresponding 2D system described by spherical coordinates $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, with fixed radius $R > 0$, which is also assumed to be the space unit. In the dimensional reduction, δR is also taken as being the radial thickness of the spherical bubble. The two-dimensional approximation is reasonable since excitations in the radial direction are inaccessible due to the large amount of energy required for them. This is true when the thickness δR of the 3D spherical shell is very small compared to its radius R , that is, $\delta R \ll R$, as discussed in Refs.[15, 22]. We also stress that our main concern is the *dynamic stability* of the dark soliton. We do not take into account its *energetic instability*, which could be triggered by a thermal cloud that is neglected here. The condensate can be described in the mean-field approach as a system of two coupled GPEs [38, 39] with the nonlinear two-body parameter $\bar{g} \equiv 4\pi\hbar^2 aN/M$, where N is the number of atoms with mass M , and a is the atom-atom s -wave scattering length. With this definition for the two-body contact

interaction, the corresponding wave function is assumed normalized to one in the L^2 -norm. By giving R as the length unit, with the time in units of $MR^2/(2\hbar)$, the dimensionless 2D GPE for the wave function $\psi \equiv \psi(\theta, \phi, t)$ is written as

$$\begin{aligned} i\partial_t \psi &= -\Delta_{2D} \psi + g|\psi|^2 \psi \\ &\equiv -\left[\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right] \psi + g|\psi|^2 \psi, \end{aligned} \quad (1)$$

where $g \equiv 4\sqrt{2\pi}aN/\delta R$. With ℓ and m being the quantum numbers related to the angular variables ϕ and θ , a given state ψ is given by $\psi_{\ell m}$. By separating the m -angular mode, with $\psi_{\ell m} \equiv e^{im\phi} \Theta_{\ell m}(\theta)$, the above defined Δ_{2D} is reduced to the 1D operators Δ_m , leading to the Laplace equation

$$-\Delta_m \Theta_{\ell m} = -\left[\frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} - \frac{m^2}{\sin^2 \theta} \right] \Theta_{\ell m} = \lambda_\ell \Theta_{\ell m}. \quad (2)$$

Recall that bounded solutions of this equation exist if and only if $\lambda_\ell = \ell(\ell+1)$, with $\ell \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, and $-\ell \leq m \leq \ell$. They can be expressed by the associate Legendre polynomials, with $\Theta_{\ell m}(\theta) = P_\ell^m(\theta)$.

Existence of dark solitons — The existence of dark solitons on a bubble (dark ring solitons), previously investigated in [33, 34], can be associated with the stationary solution of the GPE,

$$-\left(\frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} \right) f(\theta) + \varepsilon |f(\theta)|^2 f(\theta) = \mu f(\theta), \quad (3)$$

where $\varepsilon = g/(2\pi)$ is the magnitude of the defocusing nonlinearity, and μ is the chemical potential. To obtain (3) from (1), we assume

$$\psi_s(\theta, \phi, t) = \frac{f(\theta)}{\sqrt{2\pi}} e^{-i\mu t}, \quad \int_0^\pi \sin \theta |f(\theta)|^2 d\theta = 1. \quad (4)$$

Eq. (3) admits the conservation of flux,

$$J = \sin \theta \left[\bar{f}(\theta) \frac{df(\theta)}{d\theta} - \frac{d\bar{f}(\theta)}{d\theta} f(\theta) \right], \quad 0 < \theta < \pi, \quad (5)$$

where $f(\theta)$, $df(\theta)/d\theta$ are bounded at zeros of $\sin \theta$. Hence, we have $J = 0$ and, without loss of generality, we can consider a real-valued function $f(\theta) : [0, \pi] \rightarrow \mathbb{R}$. The dark soliton profile is defined by a density that decreases monotonically, $|f(\theta)|^2$ on $[0, \pi]$, with zero at $\theta = \frac{\pi}{2}$. Given the normalization (4), μ is defined by ε , as

$$\mu = \int_0^\pi \sin \theta \left(\left| \frac{df(\theta)}{d\theta} \right|^2 + \varepsilon |f(\theta)|^4 \right) d\theta. \quad (6)$$

A constant solution of (3) exists for every $\varepsilon > 0$ in the form: $f(\theta) = 1/\sqrt{2}$, with $\mu = \varepsilon/2$. A dark soliton is obtained by the small ε limiting solution of (3):

$$f(\theta) = \sqrt{\frac{3}{2}} \cos \theta + \varepsilon f_1(\theta) + \mathcal{O}(\varepsilon^2), \quad \mu = 2 + \frac{9}{10} \varepsilon + \mathcal{O}(\varepsilon^2), \quad (7)$$

where $f_1(\theta)$ is uniquely defined and proportional to $P_3(\cos \theta)$ [see (S9) in the SM [37]]. As the main term

of (7) is the normalized $P_1(\cos \theta)$, the orthogonality of the Legendre polynomials imply that the normalization (4) for small ε is restricted to $\mathcal{O}(\varepsilon^2)$.

In the other extreme, with $\varepsilon \rightarrow \infty$, the dark soliton profile becomes concentrated near the equator ($\theta = \pi/2$), according to the exact solution in the flat space:

$$-\frac{d^2}{d\theta^2}f_\infty(\theta) + \varepsilon f_\infty^3(\theta) = \mu_\infty(\varepsilon)f_\infty(\theta), \quad (8)$$

$$f_\infty(\theta) = \frac{1}{\sqrt{2}} \tanh \left[\frac{\sqrt{\varepsilon}}{2} \left(\frac{\pi}{2} - \theta \right) \right], \quad \mu_\infty(\varepsilon) = \frac{\varepsilon}{2},$$

With the convenient variable change $z \equiv \frac{\sqrt{\varepsilon}}{2} \left(\frac{\pi}{2} - \theta \right)$, notice that $f_\infty(\theta)$ is normalized to one only if $\sqrt{\varepsilon} \rightarrow \infty$. It can be incorporated into the asymptotic expansion, as shown in [37],

$$f(\theta) = f_\infty(\theta) + \frac{f_\infty^{(1)}(\theta)}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right), \quad \mu = \frac{\varepsilon}{2} + \sqrt{\varepsilon} + \mathcal{O}(1), \quad (9)$$

with $f_\infty^{(1)}(\theta)$ being uniquely defined, given by (S17).

In Fig. 1(a) we present profiles of the dark soliton obtained numerically, by using the shooting method described in [37]. The profile $f(\theta)$ is monotonically decreasing with the only zero at $\theta = \pi/2$ (the equator). It is close to $\sim \cos \theta$ as $\varepsilon \rightarrow 0$ according to (7) and close to $f_\infty(\theta)$ as $\varepsilon \rightarrow \infty$ according to (9)–(9). We confirm the asymptotic predictions by plotting μ versus ε in Fig. 1(b), which shows a good agreement between the theory and numerical results.

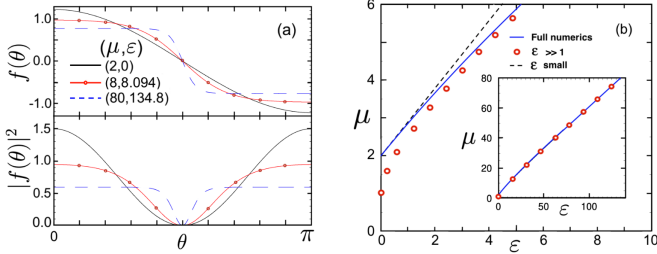


FIG. 1. (Color on-line) In (a), we show profiles of dark solitons, $f(\theta)$ (upper panel), with corresponding densities $|f(\theta)|^2$ (lower panel), as function of θ , for three different values of μ , with ε obtained numerically by using the shooting method. In (b), the chemical potential μ is presented as a function of ε for dark solitons. Solid lines correspond to the numerical data, with dashed and circles referring to $\mu = 2 + \frac{9}{10}\varepsilon$ and $\mu = \frac{\varepsilon}{2} + \sqrt{\varepsilon} + 1$, respectively, in agreement with (7) and (9).

Stability of dark solitons — Linear stability analysis [40] can be performed to verify the stability of dark solitons. In this context, the well-known Bogoliubov-de Gennes (BdG) method was considered in [15]. Within this approach, small amplitude oscillations are assumed around the stationary solution (4), which is redefined as

$$\psi(\theta, \phi, t) = \psi_s(\theta, \phi, t) + \frac{e^{-i\mu t}}{\sqrt{2\pi}} [u(\theta, \phi, t) + iv(\theta, \phi, t)], \quad (10)$$

with u and v being real-valued. From Eqs. (1), (4), and (10), we obtain the linearized equation of motion, which leads to the following coupled equations for u and v :

$$\begin{cases} \partial_t u = -\Delta_{2D} v + \varepsilon f(\theta)^2 v - \mu v, \\ -\partial_t v = -\Delta_{2D} u + 3\varepsilon f(\theta)^2 u - \mu u, \end{cases} \quad (11)$$

where the chemical potential μ is a function of $\varepsilon = g/(2\pi)$, given by (6). Performing the separation of variables for m -angular modes in ϕ , with (u, v) given by the eigenvectors $(\hat{u}_m, \hat{v}_m)$ and eigenvalue ω ,

$$u = \hat{u}_m(\theta)e^{i(m\phi + \omega t)}, \quad v = \hat{v}_m(\theta)e^{i(m\phi + \omega t)}, \quad (12)$$

we define the spectral stability problem. For each m -mode separately, we have the coupled system

$$\begin{cases} \omega \hat{u}_m = L_m^- \hat{v}_m, & L_m^- = -\Delta_m + \varepsilon f(\theta)^2 - \mu, \\ \omega \hat{v}_m = L_m^+ \hat{u}_m, & L_m^+ = -\Delta_m + 3\varepsilon f(\theta)^2 - \mu, \end{cases} \quad (13)$$

with Δ_m given in (2).

The dark soliton is spectrally stable with respect to the m -angular mode if the corresponding imaginary part of ω is zero, for all eigenvalues of (13). Let us consider, for each mode m , the corresponding ω_m . As we know from the general theory [40], all eigenvalues ω_m of the spectral stability problem (13) are real if the linear operators L_m^- and L_m^+ are positive. These linear operators enjoy two relations for applications of the comparison principle:

$$L_m^+ - L_m^- = 2\varepsilon f^2(\theta) \geq 0 \quad (14)$$

and

$$L_{m+1}^\pm - L_m^\pm = \frac{2m+1}{\sin^2 \theta} \geq 0. \quad (15)$$

Based on (14), in [37] we demonstrate that all eigenvalues of $L_m^\pm$ are strictly positive for $m \geq 2$ and small $\varepsilon > 0$, such that all eigenvalues of L_m^+ remain strictly positive for $m \geq 2$ and all $\varepsilon > 0$. Hence, the dark soliton is stable with respect to the m -angular mode for small ε . However, the lowest eigenvalue of L_m^- , defined as ω_m^- , can cross zero and become negative for sufficiently large ε , triggering the snaking instability of the dark solitons.

Next, by considering the comparative relation (15), we demonstrate in [37] that the smallest eigenvalue of L_{m+1}^- crosses zero for values of ε larger than the smallest eigenvalue of L_m^- for any given $m \geq 2$, which always occurs for large values of m . This implies that, for every $m \geq 2$, there exists a $\varepsilon_m > 0$ satisfying $\varepsilon_m < \varepsilon_{m+1}$, such that L_m^- has a simple negative eigenvalue for $\varepsilon > \varepsilon_m$. This further clarifies that the spectral stability problem (13) provides a real unstable eigenvalue m for $\varepsilon > \varepsilon_m$. As shown in Table I, the asymptotic expression, derived in [37] for $m \gg 2$,

$$\varepsilon_m^{th} = 4m(m-1), \quad (16)$$

provides a good approximation to predict the threshold onset of instability, considering all values of $m \geq 2$. The

TABLE I. For each angular mode m , the threshold values [where $\text{Im}(\omega_m) = 0$] of ε are given, with ε_m being the exact numerical results and ε_m^{th} given by the analytical approximation (16). The respective values of μ are also presented.

m	2	3	4	5	6	7
ε_m	8.367	24.402	48.416	80.420	120.420	168.420
ε_m^{th}	8	24	48	80	120	168
μ	8.182	18.202	32.208	50.210	72.210	98.210

$\mathcal{O}(1)$ discrepancy between (16) and the numerical data represents less than 5% even for $m = 2$.

The analytical predictions are confirmed numerically, as shown in Fig. 2 for the angular modes with $m = 2, 3, 4$, and 5. The onset of instability in the spectral problem (13) corresponds to the zero eigenvalues of the linear operator L_m^- , with the imaginary part of the frequencies dynamically characterizing the instability. The dark soliton becomes unstable when $\varepsilon \gtrsim 8.37$, with $m = 2$ being the initially dominant mode. The $m = 2$ mode remains dominant for larger ε , until $\varepsilon \approx 35$ [when $\text{Im}(\omega_2) = \text{Im}(\omega_3)$], where the mode $m = 3$ starts dominating the instability. As represented in Fig. 2, for a given ε , the largest imaginary eigenvalue characterizes the dominant unstable mode. If a given ε corresponds to a dominant mode $m \geq 2$, then the dark soliton is affected by snake instability and tends to break into m vortex-antivortex pairs.

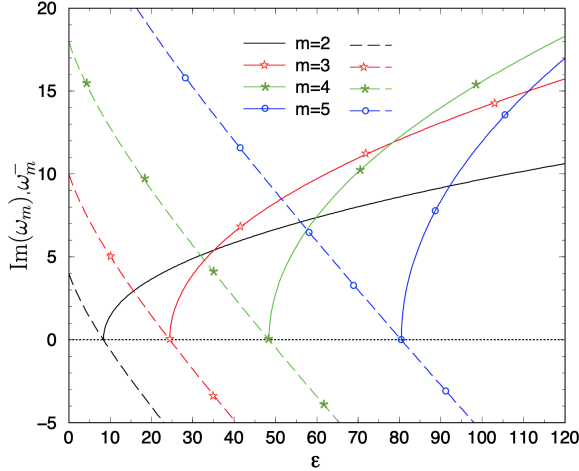


FIG. 2. (Color on-line) By varying ε , we show for different m -angular modes (indicated by the symbols and lines) that the imaginary part of the eigenvalues ω (positive defined), obtained from (13) (solid lines), start increasing from zero when the corresponding lowest eigenvalue ω_m^- of L_m^- (dashed lines) becomes negative. For a given ε , the dominant unstable mode m is provided by the largest $\text{Im}(\omega_m)$.

Regarding the spectral stability of dark solitons for $m = 1$, we demonstrate in [37] that the Eq. (13) has only real eigenvalues ω , with no instability bifurcations. Finally, for $m = 0$, it is also shown in [37] that there exists a single pair of eigenvalues ω of negative energy, which are

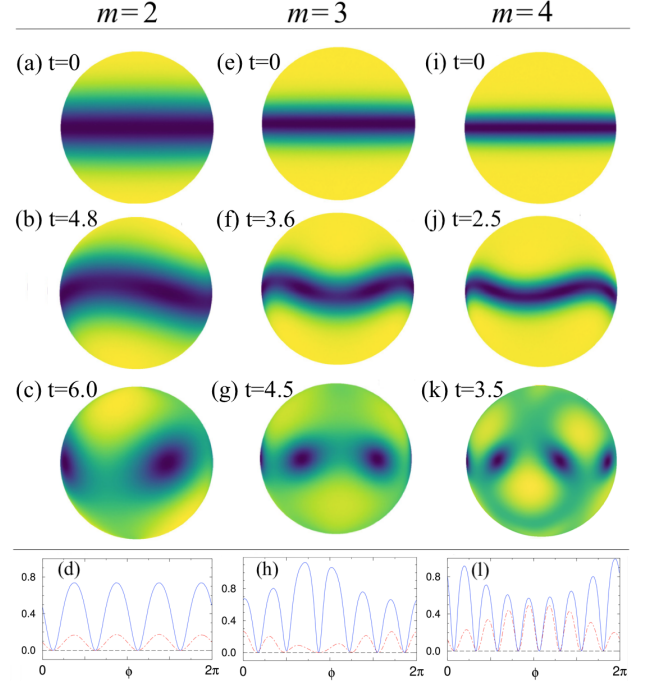


FIG. 3. Dark-soliton dynamics, for given time instants t , with dominant instability modes $m = 2$ (left panels, $\varepsilon = 20$), $m = 3$ (center panels, $\varepsilon = 50$), and $m = 4$ (right panels, $\varepsilon = 100$), shown by the respective densities $|\psi|^2$. In 3D graphics, the darker the region, the less dense it is. Panels (d), (h), and (l) show the corresponding $|\psi(\frac{\pi}{2}, \phi)|^2$ for the same instants t represented in the upper 3D plots. The dashed lines are for $t = 0$; the red dotted-dashed, when snake instabilities occur; and the solid-blue lines, after the breakup in vortex pairs.

smaller than all other pairs of eigenvalues ω of positive energy for small values of ε . These pairs can coalesce hypothetically for large values of ε , triggering another instability bifurcation. However, we have no supporting numerical evidence for the occurrence of such instability.

In Fig. 3 we illustrate the dynamical instability of the dark soliton in a sphere for three different dominant modes: $m = 2$, $m = 3$, and $m = 4$. The three choices are taken respectively at $\varepsilon = 20, 50$, and 100 [See Fig. 2 for the given data]. At $t = 0$, we observe the dark-soliton positions at the equator, with their respective widths being smaller for larger values of ε . Next, in the time evolution, we can verify the onset of snake instabilities provoking the breakup of such dark solitons in vortex-antivortex pairs. As seen in panels (c) and (d), for $m = 2$, two pairs of vortices are formed; in (g) and (h), for $m = 3$, we observe the formation of three pairs of vortices; and, in panels (k) and (l), for $m = 4$, we have four pairs of vortices. The widths of the dark solitons, as well as the healing lengths of the generated vortices (after the breakup of the snake instabilities), correspond to $\sim 1/\sqrt{\varepsilon} \approx 0.22$ ($m = 2$), 0.14 ($m = 3$), and 0.10 ($m = 4$).

The topology of the sphere implies that, on the onset of instability, the vortex number change must be $+2$, due

to Poincaré-Hopf theorem, which states that any continuous tangent vector field on a sphere must have at least one point where it is zero. For a superfluid, this implies no single vortex on a sphere. They must exist in pairs of opposite circulation (vortex-antivortex pairs) whose charges sum to zero.

The numerical approach considered in the dark-soliton dynamics is carried out first by obtaining the stationary solutions using the shooting method [48, 49]. The time evolution employs the combined spectral method together with the finite difference method introduced in [15]. For the full numerical calculations of the GPE (1), we have assumed a time-step $\delta t = 10^{-5}$, with spatial grids in θ and ϕ directions of sizes 256×256 . The respective step sizes were $\delta\theta = \pi/256 \approx 0.0123$ and $\delta\phi = 2\pi/256 \approx 0.0245$. Details of the numerical methods are given in [37].

Conclusions — In the present letter, we have reported results obtained numerically and analytically for the dynamical stability of dark solitons in a single atomic BEC trapped on the surface of a rigid spherical bubble with radius R and thickness δR . All the numerical and analytical results rely on a single parameter ε , which encompasses all information on the repulsive two-body interaction, the number of atoms, and the thickness of the bubble surface. The results of our analysis, in terms of the azimuthal excitation angular momentum modes m , should be useful when considering possible experimental realizations. We expect these results to impact investigations of BEC systems in spherical geometries, which are being performed aboard the international space station [2], as well as in shell condensates made with atomic mixtures [6]. Based on the spectral stability analysis, our study in the sphere shows that for the interaction parameter $\varepsilon \lesssim 8.37$, dark solitons are stable for all m -angular mode excitations. For $\varepsilon \gtrsim 8.37$, modes with $m \geq 2$ are excited, where the dominant m -angular mode induces snake-like instabilities that cause the soliton to break up with the production of m -vortex pairs. In summary, we are highlighting the occurrence of a phenomenon that has not been previously reported when considering predictions from purely analytical studies compared with exact computational results. Our predictions for a single BEC in a sphere should provide valuable information for ongoing experimental investigations.

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**STABILITY OF DARK SOLITONS IN A BUBBLE
BOSE-EINSTEIN CONDENSATE -
SUPPLEMENTAL MATERIAL**

Dark solitons for small ε

Let us consider the profile $f(\theta) : [0, \pi] \rightarrow \mathbb{R}$, bounded at the end points $[\theta = 0, \pi]$, defined from the nonlinear differential equation,

$$-\left[\frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} - \varepsilon f^2(\theta)\right] f(\theta) = \mu f(\theta), \quad (\text{S1})$$

where $\varepsilon > 0$ is a small parameter for the defocusing non-linearity. If we impose the normalization constraint

$$\int_0^\pi d\theta \sin \theta |f(\theta)|^2 = 1, \quad (\text{S2})$$

then there exists a countable set of solutions $\{\mu_\ell\}_{\ell \in \mathbb{N}_0}$, uniquely parameterized by $\varepsilon > 0$, which bifurcate from the linear modes $\Theta_\ell(\theta) = P_\ell(\cos \theta)$ of the Laplace equation $-\left[\frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta}\right] \Theta_\ell = \ell(\ell+1)\Theta_\ell$, where $P_\ell(\cos \theta)$ is the Legendre polynomial of degree ℓ . Therefore, for small ε in (S1), we obtain

$$f(\theta) = \frac{P_\ell(\cos \theta)}{\left[\int_0^\pi d\theta \sin \theta |P_\ell(\cos \theta)|^2\right]^{1/2}} + \mathcal{O}(\varepsilon), \quad (\text{S3})$$

with μ having the following dependence on ε :

$$\mu(\varepsilon) = \ell(\ell+1) + \varepsilon \frac{\int_0^\pi d\theta \sin \theta |P_\ell(\cos \theta)|^4}{\left[\int_0^\pi d\theta \sin \theta |P_\ell(\cos \theta)|^2\right]^2} + \mathcal{O}(\varepsilon^2). \quad (\text{S4})$$

For $\ell = 0$, we have the trivial constant solution

$$f(\theta) = \frac{1}{\sqrt{2}}, \quad \mu(\varepsilon) = \frac{\varepsilon}{2}. \quad (\text{S5})$$

For $\ell = 1$, we have a dark-soliton solution, which can be derived, for small expansion in ε , from (S3) and (S4), as

$$f(\theta) = \sqrt{\frac{3}{2}} \cos \theta + \varepsilon f_1(\theta) + \mathcal{O}(\varepsilon^2), \quad (\text{S6})$$

$$\mu(\varepsilon) = 2 + \frac{9}{10} \varepsilon + \mathcal{O}(\varepsilon^2). \quad (\text{S7})$$

The profile $f(\theta)$ is monotonically decreasing for every $\varepsilon > 0$. In this case, we obtain

$$-\left[\frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} + 2\right] f_1(\theta) = \frac{3}{10} \sqrt{\frac{3}{2}} \cos \theta (3 - 5 \cos^2 \theta). \quad (\text{S8})$$

Under the normalization (S2), a unique solution $f_1(\theta)$ is possible to solve the linear inhomogeneous equation (S8). It can be obtained explicitly, using the orthogonality condition $\int_0^\pi d\theta \sin \theta \cos \theta f_1(\theta) = 0$ which follows from (S2), being proportional to $P_3(\cos \theta)$. So, from the Laplace equation, it satisfies $-f_1''(\theta) - \cot \theta f_1'(\theta) = 12 f_1(\theta)$, (using primes for derivatives). Its exact expression is

$$f_1(\theta) = \frac{3}{100} \sqrt{\frac{3}{2}} \cos \theta (3 - 5 \cos^2 \theta). \quad (\text{S9})$$

By substituting in (S6), the solution for small ε is

$$f(\theta) = \sqrt{\frac{3}{2}} \cos \theta \left[1 + \frac{3\varepsilon}{100} (3 - 5 \cos^2 \theta) + \mathcal{O}(\varepsilon^2)\right]. \quad (\text{S10})$$

Dark solitons for large ε

We recall that the dark soliton profile $f(\theta)$ vanishes at $\theta = \frac{\pi}{2}$. As ε increases ($\varepsilon \gg 1$), its reduction becomes concentrated near $\theta = \frac{\pi}{2}$. The asymptotic solution is

$$-f_\infty''(\theta) + \varepsilon f_\infty^3(\theta) = \mu_\infty(\varepsilon) f_\infty(\theta). \quad (\text{S11})$$

By connecting it with the constant solution (S5), and redefining it as $f_\infty(\theta) \equiv g_0(z)$, with $z \equiv \frac{\sqrt{\varepsilon}}{2} (\frac{\pi}{2} - \theta)$, we have $g_0(z) = \frac{1}{\sqrt{2}} \tanh(z)$ as an exact solution of (S11):

$$-\frac{\varepsilon}{4} g_0''(z) + \varepsilon g_0^3(z) = \frac{\varepsilon}{2} g_0(z) = \mu_\infty(\varepsilon) g_0(z), \quad (\text{S12})$$

where $\frac{d}{d\theta} = -\frac{\sqrt{\varepsilon}}{2} \frac{d}{dz}$. Further, with $f(\theta) \equiv g(z)$, the original equation (S1) can be written as

$$-\frac{\varepsilon}{4} g''(z) + \frac{\sqrt{\varepsilon}}{2} \tan\left(\frac{2z}{\sqrt{\varepsilon}}\right) g'(z) + \varepsilon g^3(z) = \mu g(z). \quad (\text{S13})$$

A formal expansion as $\varepsilon \rightarrow \infty$ generates the term $zg'(z)$ in the lowest order. Since we need to use the normalization condition (S2), the asymptotic expansions of $g(z)$ and $\mu(\varepsilon)$, for $\varepsilon \rightarrow \infty$, are modified, such that

$$g(z) = g_0(z) + \frac{1}{\sqrt{\varepsilon}} g_1(z) + \mathcal{O}\left(\frac{1}{\varepsilon}\right), \quad (\text{S14})$$

$$\mu(\varepsilon) = \frac{\varepsilon}{2} \left[1 + \frac{2\mu_1}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right)\right]. \quad (\text{S15})$$

Substituting in (S13), we obtain the equation for $g_1(z)$:

$$-g_1''(z) + [4 - 6 \operatorname{sech}^2(z)] g_1(z) = 4\mu_1 g_0(z). \quad (\text{S16})$$

Since $\mathcal{L}_+ := -\partial_z^2 + [4 - 6 \operatorname{sech}^2(z)]$ has a kernel spanned by $\operatorname{sech}^2(z)$, and $g_0(z) = \frac{1}{\sqrt{2}} \tanh(z)$ is an odd function, a unique bounded solution exists for (S16), given by

$$g_1(z) = \frac{\mu_1}{\sqrt{2}} [\tanh(z) + z \operatorname{sech}^2(z)] = \mu_1 \frac{d}{dz} [zg_0(z)]. \quad (\text{S17})$$

Next, we can fix μ_1 using the normalization constraint (S2), together with (S14) and (S15). From the expansion

$$\begin{aligned} g^2(z) &= g_0^2(z) + \frac{2}{\sqrt{\varepsilon}} g_0(z) g_1(z) + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \\ &= g_0^2(z) \left[1 + \frac{2\mu_1}{\sqrt{\varepsilon}}\right] + \frac{\mu_1}{\sqrt{\varepsilon}} z \frac{d}{dz} g_0^2(z) + \mathcal{O}\left(\frac{1}{\varepsilon}\right), \end{aligned} \quad (\text{S18})$$

and observing that $g_0(z) = f_\infty(\theta)$ is normalized to one only asymptotically, for $\sqrt{\varepsilon} \rightarrow \infty$, such that

$$\int_0^\pi d\theta \sin \theta g_0^2(z) \Big|_{z=\frac{\sqrt{\varepsilon}}{2}(\frac{\pi}{2}-\theta)} = 1 - \frac{2}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon^3}}\right), \quad (\text{S19})$$

the normalization constraint (S2) yields

$$\begin{aligned}
1 &= \int_0^\pi d\theta \sin \theta f^2(\theta) = \frac{2}{\sqrt{\varepsilon}} \int_{-\frac{\pi\sqrt{\varepsilon}}{4}}^{\frac{\pi\sqrt{\varepsilon}}{4}} dz \cos\left(\frac{2z}{\sqrt{\varepsilon}}\right) g^2(z) \\
&= \left(1 + \frac{2\mu_1}{\sqrt{\varepsilon}}\right) \left(1 - \frac{2}{\sqrt{\varepsilon}}\right) + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \\
&+ \frac{2\mu_1}{\varepsilon} \int_{-\frac{\pi\sqrt{\varepsilon}}{4}}^{\frac{\pi\sqrt{\varepsilon}}{4}} dz \cos\left(\frac{2z}{\sqrt{\varepsilon}}\right) z \frac{d}{dz} g_0^2(z) \\
&= 1 + \frac{2(\mu_1 - 1)}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right). \tag{S20}
\end{aligned}$$

Hence, $\mu_1 = 1$ in (S14), to satisfy the (S20), implying

$$\mu(\varepsilon) = \frac{\varepsilon}{2} \left[1 + \frac{2}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \right], \quad \text{as } \varepsilon \rightarrow \infty. \tag{S21}$$

Stability analysis

For the stability analysis, the spectrum for each m -angular mode is considered separately, with the following coupled system for the operators $L_m^\pm$:

$$\begin{cases} \omega \hat{u}_m = L_m^- \hat{v}_m \equiv [-\Delta_m + \varepsilon f^2(\theta) - \mu] \hat{v}_m, \\ \omega \hat{v}_m = L_m^+ \hat{u}_m \equiv [-\Delta_m + 3\varepsilon f^2(\theta) - \mu] \hat{u}_m. \end{cases} \tag{S22}$$

where $\Delta_m = \left[\frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} - \frac{m^2}{\sin^2 \theta} \right]$. We need to determine the number of negative and zero eigenvalues of the operators $L_m^\pm$ if μ and $f(\theta)$ are defined along the branch of dark soliton solutions (S6). If $L_m^\pm$ are strictly positive, then the eigenvalues ω in the spectral stability problem (S22) are real [1], implying that the dark solitons are spectrally stable with respect to the m -angular Fourier mode. The number of unstable eigenvalues ω with $\text{Im}(\omega) \neq 0$ can be controlled by the number of negative eigenvalues of $L_m^\pm$ and the multiplicity of their zero eigenvalues (see Theorems 1.7, 1.8, and 3.10 in [1]).

About the eigenvalues of $L_m^\pm$, the following facts are applied for every integer $m \geq 0$:

- They are simple because there may be at most one bounded solution of the second-order differential equation $L_m^\pm \chi_m^\pm(\theta) = \omega_m^\pm \chi_m^\pm(\theta)$ at each endpoint of the interval $[0, \pi]$. The eigenfunctions $\chi_m^\pm(\theta)$ in the domain of $L_m^\pm$ provide the connections between the bounded solutions as $\theta \rightarrow 0$ and $\rightarrow \pi$.
- The eigenfunctions are either even or odd with respect to the midpoint $\theta = \frac{\pi}{2}$, since $f^2(\theta)$ is even about $\theta = \frac{\pi}{2}$. Hence, if the eigenvalues are simple, then the normalized eigenfunctions satisfy $\chi_m^\pm(\pi - \theta) = \pm \chi_m^\pm(\theta)$ (plus sign for even, with minus sign for odd functions).
- The smallest eigenvalue of $L_m^\pm$ is associated with the even eigenfunction about $\theta = \frac{\pi}{2}$, with the second smallest associated with the odd eigenfunction about $\theta = \frac{\pi}{2}$.

The operators $L_m^\pm$ enjoy two comparison principles:

$$L_m^+ - L_m^- = 2\varepsilon f^2(\theta) \geq 0, \tag{S23}$$

$$L_{m+1}^\pm - L_m^\pm = \frac{2m+1}{\sin^2 \theta} \geq 0. \tag{S24}$$

Since $L_m^+ > L_m^-$ for all $\theta \in [0, \pi] \setminus \{\frac{\pi}{2}\}$ and $L_{m+1}^\pm > L_m^\pm$ for all $\theta \in (0, \pi)$, the smallest eigenvalue of L_m^- is smaller than the smallest one of L_m^+ ; and the smallest eigenvalue of $L_m^\pm$ is smaller than the smallest one of $L_{m+1}^\pm$. The same holds for the second smallest eigenvalues of the same operators in the subspace of odd functions about $\theta = \frac{\pi}{2}$. The main results for the modes $m = 0$, $m = 1$, and $m \geq 2$ are presented below.

Mode $m = 0$: We prove for the spectral problem (S22) that there exists a single pair of eigenvalues ω of negative energy which are smaller than all other pairs of eigenvalues ω of positive energy for small values of ε . These pairs can coalesce hypothetically for large values of ε , triggering another instability bifurcation. However, our numerical data does not support evidence that this instability can occur for $m = 0$.

- There exists a simple zero eigenvalue of L_0^- with the eigenfunction given by the profile $f(\theta)$ of the dark soliton because $L_0^- f(\theta) = 0$ is equivalent to (S1) for every $\varepsilon > 0$.
- By Sturm's nodal theory, L_0^- has a simple negative eigenvalue since $f(\theta)$ has a single node on $[0, \pi]$. The rest of the spectrum of L_0^- is strictly positive for every $\varepsilon > 0$.
- Since the smallest eigenvalue of L_0^- in the space of odd functions about $\theta = \frac{\pi}{2}$ is located at 0, the comparison (S23) implies that L_0^+ has at most one negative eigenvalue and the second eigenvalue of L_0^+ is strictly positive for every $\varepsilon > 0$.
- Since the smallest eigenvalue of L_1^+ is 0 for every $\varepsilon > 0$ (see the case $m = 1$), the comparison (S24) for $m = 0$ implies that L_0^+ has exactly one negative eigenvalue for every $\varepsilon > 0$. See the left panel of Fig. S1 for illustration.
- For $\varepsilon = 0$, we have $L_m^\pm|_{\varepsilon=0} = -\Delta_m - 2$. By using the eigenvalues of the Laplace equation with $m = 0$, we obtain the asymptotic approximations of eigenvalues in the stability problem (S22). For small values of $\varepsilon > 0$, a special pair of simple eigenvalues exists,

$$\pm \omega_0 = \pm 2 + \mathcal{O}(\varepsilon), \tag{S25}$$

which has negative energy, since we have for the eigenvector $(\hat{u}, \hat{v})$ of (S22) with $\pm \omega_0$:

$$\langle L_0^+ \hat{u}, \hat{u} \rangle = \langle L_0^- \hat{v}, \hat{v} \rangle < 0. \tag{S26}$$

In addition, there exists the double-zero eigenvalue due to the rotational invariance and a countable sequence of pairs of simple eigenvalues

$$\pm \omega_\ell = \pm [\ell(\ell + 1) - 2] + \mathcal{O}(\varepsilon), \quad \ell \geq 2, \tag{S27}$$

with the positive energies since we have for the eigenvector $(\hat{u}, \hat{v})$ of (S22) with $\pm\omega_\ell$:

$$\langle L_0^+ \hat{u}, \hat{u} \rangle = \langle L_0^- \hat{v}, \hat{v} \rangle > 0. \quad (\text{S28})$$

- Since $\omega_\ell > \omega_0$ for every $\ell \geq 2$, there is some $\varepsilon_0 > 0$ (which might be infinite) such that the stability problem (S22) for $m = 0$ and $\varepsilon \in (0, \varepsilon_0)$ admits only real eigenvalues ω .
- The only way to get complex unstable eigenvalues ω of the stability problem (S22) for $\varepsilon \geq \varepsilon_0$ is under the condition that the eigenvalues $\pm\omega_0$ of negative energy coalesce with eigenvalues $\{\pm\omega_\ell\}_{\ell=2}^\infty$ of positive energy for some $\varepsilon = \varepsilon_0$. Numerical evidence shows that this does not occur and $\varepsilon_0 = \infty$.

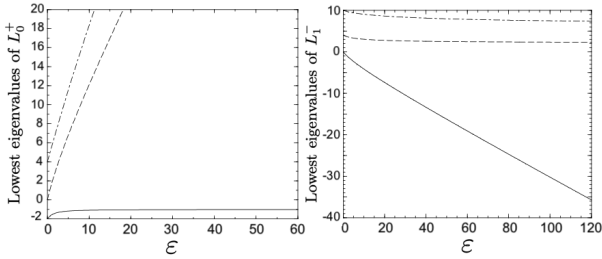


FIG. S1. Three smallest eigenvalues of L_0^+ (left) and three smallest eigenvalues of L_1^- (right) versus ε . The eigenvalues were calculated numerically by discretizing the operators with finite differences up to 800 points.

Mode $m = 1$: We prove for the spectral problem (S22) that only real eigenvalues ω exist for every $\varepsilon > 0$.

- A simple zero eigenvalue of L_1^+ exists, with the eigenfunction $f'(\theta)$, as $L_1^+ f'(\theta) = 0$ is equivalent to differentiating (S1) with respect to θ for every $\varepsilon > 0$.
- By Sturm's nodal theory, L_1^+ does not have negative eigenvalues. $f'(\theta) < 0$ is sign-definite for $\theta \in (0, \pi)$ due to monotonicity of the dark soliton profile (S6). So, L_1^+ admits a simple zero eigenvalue, with the rest of its spectrum being strictly positive for all $\varepsilon > 0$.
- The comparison (S23) implies that L_1^- has at least one negative eigenvalue for every $\varepsilon > 0$.
- Since the smallest eigenvalue of L_0^- in the space of odd functions about $\theta = \frac{\pi}{2}$ is located at 0 for every $\varepsilon > 0$ (see the case $m = 0$), the comparison (S24) for $m = 0$ implies that L_1^- has exactly one negative eigenvalue and the second eigenvalue of L_1^- is strictly positive for every $\varepsilon > 0$. See the right panel of Fig. S1 for illustration.
- For $\varepsilon = 0$, we have $L_1^\pm|_{\varepsilon=0} = -\Delta_1 - 2$. By using the eigenvalues of the Laplace equation for $m = 1$, we obtain the asymptotic approximations of the eigenvalues in the stability problem (S22) for $m = 1$ and

small values of $\varepsilon > 0$. There exists a double-zero eigenvalue due to the derivative mode $f'(\theta)$ and a countable sequence of pairs of simple eigenvalues $\{\omega_\ell\}_{\ell=2}^\infty$, defined by exactly the same formula (S27).

- Since all eigenvalues have positive energy for every $\ell \geq 2$ and no change of eigenvalues of $L_1^\pm$ occurs in ε , the stability problem (S22) for $m = 1$ admits only real eigenvalues ω for every $\varepsilon > 0$.

Modes $m \geq 2$: For these cases, it is demonstrated that all eigenvalues of $L_m^\pm$ are strictly positive for small $\varepsilon > 0$ and that the eigenvalues of L_m^+ remain strictly positive for every $\varepsilon > 0$. The smallest eigenvalue of L_m^- can cross 0 at $\varepsilon = \varepsilon_m > 0$ and become negative for $\varepsilon > \varepsilon_m$. If this happens, the smallest eigenvalue of L_{m+1}^- crosses 0 for larger values of ε compared to the smallest eigenvalue of L_m^- , that is, $\varepsilon_m < \varepsilon_{m+1}$. Since we show that $\varepsilon_m = 4m(m-1) + \mathcal{O}(1)$ as $m \rightarrow \infty$ in (S31), this implies that the crossing at ε_m exists for every $m \geq 2$, triggering instability with exactly one unstable eigenvalue ω in the spectral stability problem (S22). These analytical results are well illustrated by the numerical results in Fig 2.

- All eigenvalues of $L_m^\pm$ are strictly positive for small $\varepsilon > 0$ since $L_m^\pm|_{\varepsilon=0} = -\Delta_m - 2$ and the eigenvalues of the Laplace equation for $m \geq 2$ yield $\ell(\ell+1) - 2 > 0$ for $\ell \geq m$. Hence, there exists $\varepsilon_0 > 0$ such that the stability problem (S22) for $m \geq 2$ and $\varepsilon \in (0, \varepsilon_0)$ admits only pairs of simple real eigenvalues $\{\omega_\ell\}_{\ell=m}^\infty$ of positive energy, defined by the same formula (S27).
- The comparison (S23) implies that, the smallest eigenvalue for L_m^- is always smaller than the smallest eigenvalue for L_m^+ . Since the smallest eigenvalue of L_1^+ is 0 for every $\varepsilon > 0$ (see the case $m = 1$), the comparison (S24) for $m \geq 1$ implies that the smallest eigenvalue of L_m^+ for $m \geq 2$ is strictly positive for every $\varepsilon > 0$.
- Since the smallest eigenvalue of L_1^- in the space of odd functions about $\theta = \frac{\pi}{2}$ is strictly positive for every $\varepsilon > 0$ (see the case $m = 1$), the comparison (S24) with $m \geq 1$ implies that the second eigenvalue of L_m^- for $m \geq 2$ is strictly positive for every $\varepsilon > 0$.
- The complex eigenvalues ω in the spectral stability problem (S22) may arise if and only if the smallest positive eigenvalue of L_m^- crosses 0 at $\varepsilon = \varepsilon_m$. The comparison (S24) for $m \geq 2$ implies that the smallest eigenvalue of L_m^- always cross 0 for smaller values of ε compared to the smallest eigenvalue of L_{m+1}^- so that $\varepsilon_m < \varepsilon_{m+1}$ for $m \geq 2$.

Asymptotic formula for ε_m as $m \rightarrow \infty$

By using the asymptotic solution (S14) with $\mu_1 = 1$, together with (S18), and assuming $m^2 = \mathcal{O}(\varepsilon)$ as $\varepsilon \rightarrow \infty$,

$$\begin{aligned} L_m^- &= -\frac{\varepsilon}{4}\partial_z^2 + \tan\left(\frac{2z}{\sqrt{\varepsilon}}\right)\frac{\sqrt{\varepsilon}}{2}\partial_z + \frac{m^2}{\cos^2\left(\frac{2z}{\sqrt{\varepsilon}}\right)} + \varepsilon g^2(z) - \mu \\ &= -\frac{\varepsilon}{4}\left\{\partial_z^2 - \frac{4m^2}{\varepsilon} + 2\text{sech}^2(z)\left[1 + \frac{2-2z\tanh(z)}{\sqrt{\varepsilon}}\right] \right. \\ &\quad \left. + \mathcal{O}\left(\frac{1}{\varepsilon}\right)\right\}. \end{aligned} \quad (\text{S29})$$

The spectrum of $\mathcal{L}_- = -\partial_z^2 - 2\text{sech}^2(z)$ includes a simple negative eigenvalue at -1 with the eigenfunction spanned by $\text{sech}(z)$ and the continuous spectrum at $[0, \infty)$. By using perturbation theory for an isolated eigenvalue, we obtain that L_m^- has a zero eigenvalue if and only if

$$\begin{aligned} \frac{4m^2}{\varepsilon} &= 1 + \frac{4}{\sqrt{\varepsilon}} \frac{\int_{\mathbb{R}} (1 - z \tanh(z)) \text{sech}^4(z) dz}{\int_{\mathbb{R}} \text{sech}^2(z) dz} + \mathcal{O}\left(\frac{1}{\varepsilon}\right) \\ &= 1 + \frac{2}{\sqrt{\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon}\right), \end{aligned} \quad (\text{S30})$$

which yields the quantization formula for bifurcations at $\{\varepsilon_m\}_{m=2}^\infty$. From the above, $4m^2 + 1 = (\sqrt{\varepsilon_m} + 1)^2$, or

$$\sqrt{\varepsilon_m} + 1 = 2m\sqrt{1 + \frac{1}{4m^2}} + \mathcal{O}(1),$$

which leads to

$$\varepsilon_m \approx 4m(m-1) + \mathcal{O}(1), \quad \text{as } m \rightarrow \infty. \quad (\text{S31})$$

In view of the inequality $0 < \varepsilon_m < \varepsilon_{m+1}$ for $m \geq 2$, the asymptotic formula (S31) implies that the lowest eigenvalue of L_m^- crosses 0 at $\varepsilon = \varepsilon_m$ for every $m \geq 2$.

Numerical method for dark solitons

We define $\tilde{f} \equiv \tilde{f}(\theta) = f(\theta)\sqrt{\varepsilon}$ for solutions of (S1),

$$\mu \tilde{f} = -\frac{d^2 \tilde{f}}{d\theta^2} - \cot(\theta) \frac{d\tilde{f}}{d\theta} + \tilde{f}^3, \quad (\text{S32})$$

subject to the normalization

$$\int_0^\pi d\theta \cos(\theta) |\tilde{f}(\theta)|^2 = \varepsilon. \quad (\text{S33})$$

The problem is reformulated as for a given μ , we obtain the solution $\tilde{f}(\theta)$ from (S32) and define ε from (S33). Since we are looking for dark solitons, we solve (S32) just from $\theta = 0$ to $\theta = \pi/2$ and then take the odd continuation of $\tilde{f}(\theta)$ from $\theta = \pi/2$ to $\theta = \pi$. It is a two-point boundary-value problem with the boundary conditions $\tilde{f}'(0) = 0$ and $\tilde{f}(\pi/2) = 0$.

The boundary-value problem is solved by the shooting method combined with the secant method [2]. In this

case, for a given μ we shoot two close values of $\tilde{f}(0)$ and propagate Eq. (S32) with the Runge-Kutta method till $\theta = \pi/2$. From the values obtained of $\tilde{f}(\pi/2)$ we can estimate a new initial shot by the secant method until we get $\tilde{f}(\pi/2) = 0$.

Results for $\mu = 2.2$ and 3.4 are shown in the Fig. S2. Sweeping μ from $2 < \mu \leq 20$ can be done by continuation [3], which is performed as follows. Once we get a solution $\tilde{f}_\mu(0)$ for a given μ , we increment μ by $\delta\mu = 0.1$ and use $\tilde{f}_\mu(0)$ as an initial ansatz, and after shooting combined with the secant method, we obtain $\tilde{f}_{\mu+\delta\mu}(0)$. Subsequently, we keep incrementing μ by $\delta\mu$ and take the previous μ initial value. For $\mu > 20$, continuation is not effective. For larger values of μ , we observe from (S12) that a good approximation is given by $\tilde{f}(0) = \sqrt{\mu}$. By using this initial ansatz for each μ , it was possible to obtain the dark soliton solutions from $\mu = 2$ to $\mu = 80$. The value of ε is obtained *a posteriori* by Eq. (S33).

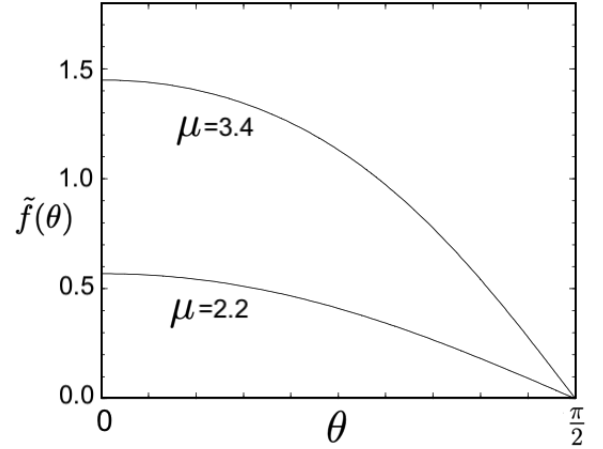


FIG. S2. Shooting method for dark solitons. The left boundary condition is $\tilde{f}'(0) = 0$. For fixed μ one shoots $\tilde{f}(0)$ and propagate Eq. (S32) till the condition $\tilde{f}(\pi/2) = 0$ is satisfied.

Numerical method for the GPE

We consider the time-dependent evolution of the GPE for $\psi \equiv \psi(\theta, \phi, t)$, as in [4], with linear part given by

$$i \frac{\partial \psi}{\partial t} = - \left[\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right] \psi. \quad (\text{S34})$$

To avoid problems at $\theta = \{0, \pi\}$, the ψ is expanded in its Fourier modes, as

$$\psi(\theta, \phi, t) = \sum_k e^{ik\phi} \psi_k(\theta, t). \quad (\text{S35})$$

A grid of $M + 1$ points is defined, with $\theta_j = jh, j \in 0, 1, \dots, M$, where $h = M/\pi$. For each $\theta_j, j = 1, \dots, M-1$,

ψ_k is computed by using a fast Fourier transform (FFT) algorithm in the ϕ direction. In the poles, where $\theta = \{0, \pi\}$ and there is no dependence on ϕ , only the mode $k = 0$ contributes. By substituting (S35) into (S34), we must solve the equation for $\psi_k \equiv \psi_k(\theta, t)$, given by

$$i \frac{\partial \psi_k}{\partial t} = - \left[\frac{\partial^2 \psi_k}{\partial \theta^2} + \cot \theta \frac{\partial \psi_k}{\partial \theta} - \frac{k^2}{\sin^2 \theta} \psi_k \right],$$

in the interval $\theta \in (0, \pi)$ with boundary conditions $\psi_k(0, t) = \psi_k(\pi, t) = 0, \forall k \neq 0$. For $k = 0$, it is required the Neumann boundary condition, $\left. \frac{\partial \psi_0}{\partial \theta} \right|_{0, \pi} = 0$. Equation (S34) is evolved one time step δt using the finite difference Crank-Nicolson (CN) method for each k . In the case $k = 0$, we add one extra point at each boundary to implement the Neumann periodic boundary conditions. After the evolution, we perform the inverse Fourier transform in the ϕ direction. To include the nonlinear term $g|\psi|^2$, we employ the split-step operator technique. The complete scheme for one time step evolution is given by

$$\begin{aligned} & \psi(\theta, \phi, t + \delta t) \\ & \simeq e^{-\frac{ig|\psi|^2 \delta t}{2}} \text{FFT}_{\phi}^{-1} \text{CN}_{\theta, \delta t, k=0, \neq 0} \text{FFT}_{\phi} \psi(\theta, \phi, t) e^{-\frac{ig|\psi|^2 \delta t}{2}}, \end{aligned}$$

where the computational operations are performed in a sequence from right to left.

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