Rogue waves in the sine-Gordon equation

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Overview

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What is a rogue wave?

Rogue waves are gigantic waves that appear out of nowhere and then disappear without trace.

A wave is said to be rogue if its magnification, the ratio of the maximum of the rogue wave versus the mean of its background, exceeds two.
The sine-Gordon equation in laboratory coordinates \((x, t)\) is:

\[
u_{tt} - u_{xx} + \sin(u) = 0,
\]

with \(u(x, t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\). The sine-Gordon equation models:

- magnetic flux in superconducting Josephson junctions
- fermions
- stability structure in galaxies
- ribbon pendulums
- much more ...
The sine-Gordon equation in characteristic coordinates $(\xi, \eta) \in \mathbb{R}^2$ is:

$$u_{\xi \eta} = \sin(u),$$

where $x = \xi + \eta$ and $t = \xi - \eta$. 
The compatibility condition, $\chi_{xt} = \chi_{tx}$, of the following Lax pair of linear equations is the sine-Gordon equation:

$$\frac{\partial}{\partial x} \chi = \begin{bmatrix} \frac{-i\gamma}{2} + \frac{i \cos(u)}{8\gamma} & \frac{i \sin(u)}{8\gamma} - \frac{1}{4}(u_x + u_t) \\ \frac{i \sin(u)}{8\gamma} + \frac{1}{4}(u_x + u_t) & \frac{i\gamma}{2} - \frac{i \cos(u)}{8\gamma} \end{bmatrix} \chi,$$

and

$$\frac{\partial}{\partial t} \chi = \begin{bmatrix} \frac{-i\gamma}{2} - \frac{i \cos(u)}{8\gamma} & -\frac{i \sin(u)}{8\gamma} - \frac{1}{4}(u_x + u_t) \\ -\frac{i \sin(u)}{8\gamma} + \frac{1}{4}(u_x + u_t) & \frac{i\gamma}{2} + \frac{i \cos(u)}{8\gamma} \end{bmatrix} \chi.$$

Here $\gamma \in \mathbb{C}$ is a spectral parameter and $\chi = (p, q)^T$ is an eigenfunction in variables $(x, t)$ [Deconick et al, 2017].
Lax pair in characteristic coordinates

The following Lax pair is compatible for solutions of the sine-Gordon equation in characteristic coordinates:

\[
\frac{\partial}{\partial \xi} \begin{bmatrix} p \\ q \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \lambda & -u_\xi \\ u_\xi & -\lambda \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}
\]

\[
\frac{\partial}{\partial \eta} \begin{bmatrix} p \\ q \end{bmatrix} = \frac{1}{2\lambda} \begin{bmatrix} \cos(u) & \sin(u) \\ \sin(u) & -\cos(u) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}
\]

where \( \lambda \) is the spectral parameter and \( \chi = (p, q)^T \) is an eigenfunction in variables \((\xi, \eta)\). The correspondence between the spectral parameters in both frames is

\[
\lambda = -2i\gamma.
\]
Compatibility of the Lax pair is really important in the analysis of the sine-Gordon equation. Applications of this Lax system include, but are not limited to:

- Analysis of modulation stability of periodic waves
- Application of the Darboux Transformation
- Solution to initial value problem
- Algebraic methods in constructing exact solutions [Pelinovsky and Chen, 2018]
How to construct a rogue wave

The one-fold Darboux Transformation (DT) creates new solutions to the sine-Gordon equation with a solution $u$ to the sine-Gordon equation and solution $(\lambda_1, p_1, q_1)$ to the Lax pair,

$$-\hat{u}_\xi = -u_\xi + \frac{4\lambda_1 p_1 q_1}{p_1^2 + q_1^2}. $$

The two-fold DT creates new solutions to the sine-Gordon equation with a solution $u$ to the sine-Gordon equation and two solutions $(\lambda_1, p_1, q_1)$ and $(\lambda_2, p_2, q_2)$ to the Lax system,

$$-\hat{u}_\xi = -u_\xi + \frac{4(\lambda_1^2 - \lambda_2^2)[\lambda_1 p_1 q_1(p_2^2 + q_2^2) - \lambda_2 p_2 q_2(p_1^2 + q_1^2)]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1 \lambda_2[4p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)]}. $$
How to construct a rogue wave

For our purposes we want:

- $-u_\xi$ to be periodic
- $-\hat{u}_\xi$ to be a rogue wave on the periodic background $-u_\xi$
Picking The Right Eigenfunctions

Question: What solutions to the linear Lax equation will make $-\hat{u}_\xi$ a proper rogue wave?
To be a proper rogue wave we want:

- The magnification factor to exceed 2.
- $\inf_{\xi_0,\eta_0,\theta_0 \in \mathbb{R}} \sup_{x \in \mathbb{R}} | -\hat{u}_\xi(\xi, \eta) + u_\xi(\xi - \xi_0, \eta - \eta_0)e^{i\theta_0} | \to 0$ as $\eta \to \pm \infty$
- Magnification to be an isolated event

We considered traveling wave backgrounds $u(\xi, \eta) = f(\xi - \eta)$, where $f : \mathbb{R} \to \mathbb{R}$. The wave speed can be set to one because of the Lorentz transformation.
Substituting the traveling wave \( u(\xi, \eta) = f(\xi - \eta) \) into the sine-Gordon equation and then solving the resultant ordinary differential equation generates potentials

\[
u_\xi = f'(\xi - \eta) = \frac{2}{k} \text{dn}\left(\frac{\xi - \eta}{k}; k\right)
\]
called rotational waves and

\[
u_\xi = f'(\xi - \eta) = 2k \; \text{cn}(\xi - \eta; k)
\]
called librational waves. Here \( k \in (0, 1) \) is the elliptic modulus parameter for Jacobi elliptic functions.
Recall the linear equation from the Lax pair:

\[ \varphi_{\xi} = U(\lambda, u) \varphi, \quad U(\lambda, u) = \begin{pmatrix} \lambda & -u_{\xi} \\ u_{\xi} & -\lambda \end{pmatrix} \]

where \( \varphi = (p, q)^T \) is the eigenfunction.

If \( u_{\xi} \) is \( L \)-periodic in \( \xi \) then Floquet theory tells us that there exists an eigenfunction of the form:

\[ \varphi(\xi) = e^{i\mu \xi} \tilde{\varphi}(\xi) \]

where \( \tilde{\varphi}(\xi) = \tilde{\varphi}(\xi + L) \), and \( \mu \in [-\frac{\pi}{L}, \frac{\pi}{L}] \).
Re-writing The Linear Lax System

It follows from Floquet theory that $\varphi_\mu(\xi)$ is a periodic eigenfunction of the spectral problem:

$$
\begin{pmatrix}
2 \frac{d}{d\xi} + 2i\mu & f' \\
f' & -2 \frac{d}{d\xi} - 2i\mu
\end{pmatrix}
\varphi = \lambda \varphi
$$

Approximating $\frac{d}{d\xi}$ with the 12-th order finite difference matrix and discretizing the domain of the eigenfunctions over one period allows us to approximate the spectral problem as a matrix eigenvector problem in MATLAB.
Spectral Pictures for Traveling Waves

Figure: Floquet Spectrum for potential

\[ u_\xi(\xi, \eta) = \frac{2}{k} \, dn\left( \frac{\xi - \eta}{k}; k \right) \]
Spectral Pictures for Traveling Waves

Figure: Floquet Spectrum for potential
\[ u_\xi(\xi, \eta) = 2k \text{cn}(\xi - \eta; k) \]
Eigenvalues extracted from algebraic method

End points of the Floquet Spectrum can be found analytically using a certain algebraic method:

\[ \lambda_{1R} = \pm \frac{1 \pm \sqrt{1 - k^2}}{k} \]

for rotational waves and

\[ \lambda_{1L} = \pm (k \pm i\sqrt{1 - k^2}) \]

for librational waves. Here \( k \in (0, 1) \) is the elliptic modulus parameter. These eigenvalues correspond to solutions of the Lax equation \((\lambda_1, p_1, q_1)\) with potential \(-f'\) that satisfies:

\[-f' = -u_\xi = p_1^2 + q_1^2.\]
Darboux Transformation

Applying the one fold DT

\[-\hat{u}_\xi = -u_\xi + \frac{4\lambda_1 p_1 q_1}{p_1^2 + q_1^2}\]

with the rotational wave and Lax pair solution \((\lambda_1 R, p_1, q_1)\) with

\[-f' = p_1^2 + q_1^2 = -\frac{2}{k} dn\left(\frac{\xi - \eta}{k}; k\right)\]

generates

\[-\hat{u}_\xi = \pm \frac{2}{k} dn\left(\frac{\xi - \eta}{k} + K(k); k\right),\]

a translated and negated version of the rotational wave. Here

\(K(k)\) is the complete elliptic integral of the first kind. We will
need to consider new eigenfunctions in order to avoid a trivial
transformation.
New eigenfunctions for rotational waves

Consider second linearly independent eigenfunctions for rotational waves of the form

\[
\hat{p}_1 = p_1 \phi_R - \frac{q_1}{p_1^2 + q_1^2}, \\
\hat{q}_1 = q_1 \phi_R + \frac{p_1}{p_1^2 + q_1^2},
\]

where the Wronskian between \((p_1, q_1)\) and \((\hat{p}_1, \hat{q}_1)\) is normalized to 1. I have introduced the function \(\phi_R : (\xi, \eta) \rightarrow \mathbb{C}\) for rotational waves.
Applying the DT with Lax solution \((\lambda_1 R, \hat{p}_1, \hat{q}_1)\) generates algebraic solitons on the background of rotational waves in the sine-Gordon equation

\[-\hat{u}_\xi = -u_\xi + \frac{4\lambda_1 \hat{p}_1 \hat{q}_1}{\hat{p}_1^2 + \hat{q}_1^2},\]

and

\[
\lim_{|\phi_R| \to \infty} -\hat{u}_\xi = \frac{2}{k} dn\left(\frac{\xi - \eta}{k} + K(k); k\right).
\]

\(|\phi_R| \to \infty\) while \((\xi, \eta)\) moves away from a particular line \(\Omega\) in the \((\xi, \eta)\)-plane. Algebraic solitons propagate along \(\Omega\).
Figure: Algebraic solitons with $k=0.85$ (left) and $k=0.95$ (right)

$$\lambda_{1R} = \frac{1 - \sqrt{1 - k^2}}{k}$$
Algebraic Soliton Pictures

Figure: Algebraic solitons with $k=0.85$ (left) and $k=0.95$ (right)

$$\lambda_{1R} = \frac{1 + \sqrt{1 - k^2}}{k}$$
Magnification of solitons

The magnification of the algebraic solitons is

$$M := \frac{\sup_{(\xi, \eta) \in \mathbb{R}^2} |\hat{u}_\xi|}{\sup_{(\xi, \eta) \in \mathbb{R}^2} |u_\xi|} = 2 \mp \sqrt{1 - k^2}.$$ 

The two threshold is definitely surpassed for the lower sign and $k \in (0, 1)$. 
Darboux Transformation

Applying the two fold DT

\[- \hat{u}_\xi = -u_\xi + \]

\[
\frac{4(\lambda_1^2 - \lambda_2^2)[\lambda_1 p_1 q_1 (p_2^2 + q_2^2) - \lambda_2 p_2 q_2 (p_1^2 + q_1^2)]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1 \lambda_2 [4 p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)]}
\]

with the librational wave and Lax pair solutions \((\lambda_{1L}, p_1, q_1)\)
and \((\lambda_2, p_2, q_2) = (\bar{\lambda}_{1L}, \bar{p}_1, \bar{q}_1)\) with

\[p_1^2 + q_1^2 = -f' = -2kcn(\xi - \eta; k)\]

generates

\[- \hat{u}_\xi = 2k \ cn(\xi - \eta; k),\]

a negation of the librational wave potential. New eigenfunctions were also considered with the librational waves in order to avoid a trivial transformation.
New eigenfunctions

In order to avoid singularities in eigenfunctions for librational waves we will consider second linearly independent eigenfunctions of the form

\[ \hat{p}_1 = \frac{\phi_L - 1}{q_1}, \]
\[ \hat{q}_1 = \frac{\phi_L + 1}{p_1}, \]

where the Wronskian is normalized to 2. I have introduced the function \( \phi_L(\xi, \eta) \rightarrow \mathbb{C} \).
Applying the two fold DT with the librational wave and Lax pair solutions \((\lambda_1 L, \hat{p}_1, \hat{q}_1)\) and \((\lambda_2, \hat{p}_2, \hat{q}_2) = (\lambda_1 L, \tilde{\hat{p}}_1, \tilde{\hat{q}}_1)\) generates an isolated rogue wave \(-\hat{u}_\xi\) with

\[
\lim_{|\phi_L| \to \infty} -\hat{u}_\xi = 2k \, \text{cn}(\xi - \eta; k).
\]

The function \(\phi_L(\xi, \eta)\) grows linearly in \(|x| + |t|\) as \(|x| + |t| \to \infty\) for every \(k \in (0, 1)\)
Figure: Rogue Waves with $k=0.5$ (left) and $k=0.8$ (right)

\[ \lambda_{1L} = (k - i \sqrt{1 - k^2}) \]
Figure: Rogue Waves with $k=0.5$ (left) and $k=0.8$ (right)

$$\lambda_{1L} = (k + i\sqrt{1 - k^2})$$
The magnification of the rogue waves is

\[
M := \frac{\sup_{(\xi,\eta) \in \mathbb{R}^2} |\hat{u}_\xi|}{\sup_{(\xi,\eta) \in \mathbb{R}^2} |u_\xi|} = 3.
\]
The growth term $\phi$ is determined by substituting the second eigenfunctions into the Lax equations and solving a transport like equation. $\phi$ is determined by integrating from a constant $C_0 \in \mathbb{R}$ to $\xi - \eta$. The figures and analysis above were performed with $C_0 = 0$ but changing $C_0$ can alter the magnification of the rogue wave.
Modifying the growth

Figure: Magnification of the rogue wave vs $C_0$ with $k=0.5$ (left) and $k=0.8$ (right)
Concluding Remarks

By finding solutions to the linear Lax equations and then applying the Darboux transformation we were able to successfully generate rogue waves occurring in the sine-Gordon equation.
References

The End