Generalized breathers and transition fronts in time-periodic nonlinear lattices

Dmitry E. Pelinovsky (McMaster University, Canada)

joint work with Chris Chong (Bowdoin College, USA)
and Guido Schneider (University of Stuttgart, Germany)

13th Conference on Nonlinear Evolution Equations and
Wave Phenomena : Computation and Theory
Athens, Georgia, USA
14-16 April 2025

1. Breathers and Modulating pulses

True breathers exist in integrable models, e.g. the sine–Gordon equation:

$$u_{tt} - u_{xx} + \sin(u) = 0,$$

The exact breather solution is

$$u(x,t) = 4 \arctan \frac{\sqrt{1-\omega^2} \cos(\omega t)}{\omega \cosh(\sqrt{1-\omega^2} x)}, \quad 0 < \omega < 1.$$

This is the standing breather which also generates a family of traveling breathers by the Lorentz transformation:

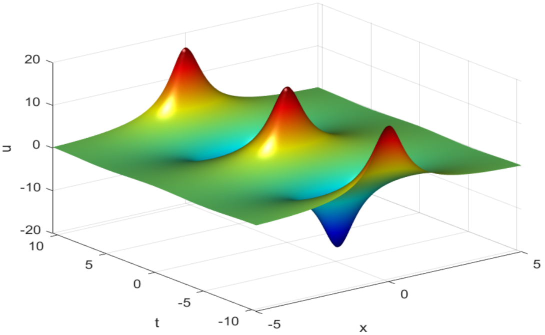
$$u(x,t) = \tilde{u} \left(\frac{x - ct}{\sqrt{1-c^2}}, \frac{t - cx}{\sqrt{1-c^2}} \right), \quad -1 < c < 1.$$

Example of the breather

The breather solution satisfies

$$u(x,t+T) = u(x,t) \quad \text{and} \quad \lim_{|x|\to\infty} u(x,t) = 0,$$

with $T = 2\pi/\omega$.



Breathers in the small-amplitude limit

Consider the limit $\omega \to 1$ in

$$u(x,t) = 4\arctan \frac{\sqrt{1-\omega^2}\cos(\omega t)}{\omega \cosh(\sqrt{1-\omega^2}x)}, \quad \omega \in (0,1),$$

where we recall that $\arctan(z) \approx z$ as $z \to 0$.

If $\varepsilon := \sqrt{1-\omega^2}$ is small, then the power expansions yields

$$u(x,t) = 4\varepsilon \text{sech}(\varepsilon x)\cos(\omega(\varepsilon)t) + \mathcal{O}(\varepsilon^3),$$

with

$$\omega(\varepsilon) = \sqrt{1-\varepsilon^2} = 1 - \frac{1}{2}\varepsilon^2 + \mathcal{O}(\varepsilon^4).$$

Small-amplitude expansions

This suggest the reduction of the sine–Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0,$$

with the small-amplitude, slow-scale expansions

$$u(x,t) = \varepsilon[A(\varepsilon x, \varepsilon^2 t)e^{it} + \bar{A}(\varepsilon x, \varepsilon^2 t)e^{-it}] + \mathcal{O}(\varepsilon^3).$$

Since $\sin(u) = u - \frac{1}{6}u^3 + \mathcal{O}(u^5)$ and $e^{\pm it}$ are in the null space of $1 + \partial_t^2$ in $L^2_{\rm per}$, we get the NLS equation for $A = A(\xi, \tau)$ from the solvability condition in $L^2_{\rm per}$ at the order of $\mathcal{O}(\varepsilon^3)$:

$$2iA_\tau - A_{\xi\xi} - \frac{1}{2}|A|^2 A = 0.$$

The breather corresponds to the NLS soliton $A(\xi, \tau) = 2\mathrm{sech}(\xi)e^{-\frac{i}{2}\tau}$.

Small-amplitude expansions

However, the expansions fail for non-integrable versions of the wave equation, e.g. for the ϕ^4 theory:

$$u_{tt} - u_{xx} + u - \frac{1}{6}u^3 = 0.$$

- H. Segur, M. D. Kruskal, Phys. Rev. Lett. 58 (1987), 747
- J. Denzler, Commun. Math. Phys. 158 (1993) 397
- B. Birnir, H.P. McKean, A. Weinstein, CPAM 47 (1994) 1043
- Justification of the NLS approximation holds only on long but finite time intervals:

$$\sup_{t\in[0,\tau_0\varepsilon^{-2}]} |u(\cdot,t) - \varepsilon A(\varepsilon\cdot,\varepsilon^2 t)e^{it} - \varepsilon\bar{A}(\varepsilon\cdot,\varepsilon^2 t)e^{-it}\|_{L^\infty} \leq C\varepsilon^3.$$

Small-amplitude expansions

The breather solutions can be thought to be a solution of the form

$$u(x,t) = v(\xi,\theta), \quad \xi := x - ct, \quad \theta := kx - \omega t$$

for some approrpriately choosen parameters c, k, ω and with boundary conditions

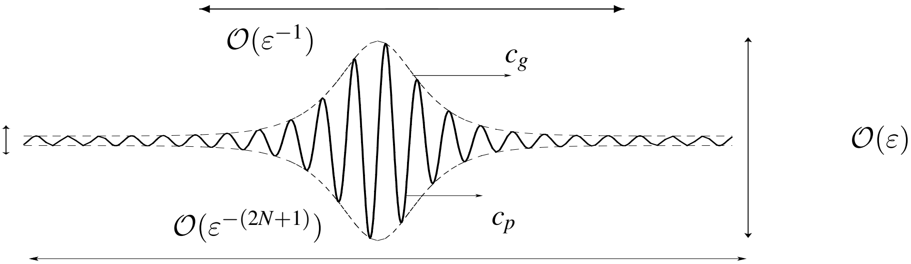
$$v(x,\theta+2\pi) = v(x,\theta) \quad \text{and} \quad \lim_{|\xi|\to\infty} v(\xi,\theta) = 0.$$

The PDE is converted to the spatial dynamical system in ξ by using Fourier series in θ. A center manifold does not allow us generally to construct a homoclinic orbit with zero boundary conditions.

- M. Groves and G. Schneider, Comm. Math. Phys. 219 (2001); J. Diff. Eqs. 219 (2005); Comm. Math. Phys. 278 (2008).

Small-amplitude expansions

In the cases with center manifolds, we have modulating pulses instead of breathers. Modulating pulses are not trully localized (also called generalized breathers or nanopterons).



Modulating pulses in spatially periodic systems

- Standing modulating pulse solutions of the wave equation:

$$u_{tt} - u_{xx} - \rho(x)u + u^3 = 0, \quad \rho(x + 2\pi) = \rho(x).$$

  with $u = v(x, kx - \omega t)$
  V. Lescarret, G. Schneider (2009); T. Dohnal, D. Rudolf (2020)

- Traveling modulating pulse solutions of the GP equation:

$$i\psi_t = -\psi_{xx} + \rho(x)\psi + |\psi|^2\psi, \quad \rho(x + 2\pi) = \rho(x)$$

  with $\psi = v(x - ct, x)e^{i\omega t}$.
  D.P & G. Schneider (2008); D.P. (2011);

- Traveling modulating pulse solutions in the wave equation:

$$u_{tt} - u_{xx} + \rho(x)u = \gamma u^3, \quad \rho(x + 2\pi) = \rho(x).$$

  with $u = v(x - ct, kx - \omega t, x)$.
  T. Dohnal, D.P., G. Schneider, Nonlinearity 37 (2024) 05505

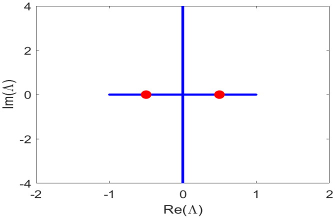
The focusing nonlinear Schrödinger (NLS) equation

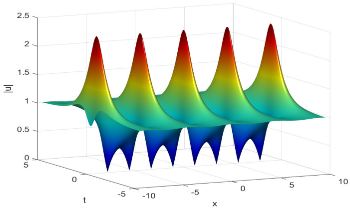
$$i\partial_t\psi + \partial_x^2\psi + |\psi|^2\psi = 0$$

admits the exact solution [Akhmediev, Eleonsky, & Kulagin (1985)]

$$\psi(x,t) = e^{it}\left[1 - \frac{2(1-\lambda^2)\cosh(k\lambda t) + ik\lambda\sinh(k\lambda t)}{\cosh(k\lambda t) - \lambda\cos(kx)}\right],$$

commonly known as Akhmediev breathers.





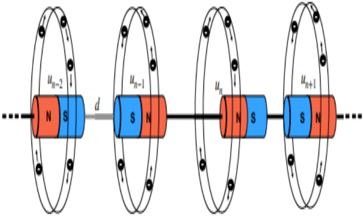
Localized-in-time breathers in time-periodic systems

The FPU model:

$$\underline{m}\ddot{u}_n + c\dot{u}_n + k(t)u_n = \beta(d + u_n - u_{n-1})^{-\alpha} - \beta(d + u_{n+1} - u_n)^{-\alpha},$$

where $\alpha, \beta, \underline{m}, d > 0$, $c \geq 0$, and $k(t + 2\pi) = k(t)$.

FPU models a chain of repelling magnets surrounded by time modulated coils (phononic lattices)
Kim, Chong, Daraios et al., Phys. Rev. E 107 (2023) 034211
Chong, Kim, Daraios et al., Phys. Rev. Res. 6 (2024) 023045



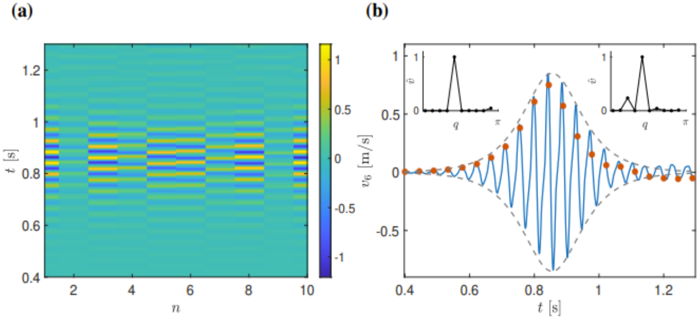
Localized-in-time breathers in time-periodic systems

The FPU model:

$$\underline{m}\ddot{u}_n + c\dot{u}_n + k(t)u_n = \beta(d + u_n - u_{n-1})^{-\alpha} - \beta(d + u_{n+1} - u_n)^{-\alpha},$$

where $\alpha, \beta, \underline{m}, d > 0$, $c \geq 0$, and $k(t + 2\pi) = k(t)$.

Localized-in-time breathers were observed in experiments:



Bifurcation theory: $c = 0$

We consider an abstract FPU system with

$$\underline{m}\ddot{u}_n + k(t)u_n = F(u_{n+1} - u_n) - F(u_n - u_{n-1}),$$

where $\underline{m} > 0$, $k(t + 2\pi) = k(t)$, and

$$F(w) = K_2 w - K_3 w^2 + K_4 w^3, \quad K_2 > 0.$$

We consider N particles with Dirichlet conditions:

$$u_0(t) = u_{N+1}(t) = 0.$$

Bifurcation theory: $c = 0$

Due to Dirichlet conditions, we use the discrete Fourier sine modes:

$$u_n(t) = \sum_{m=1}^{N} \hat{u}_m(t) \sin(q_m n), \quad q_m := \frac{\pi m}{N+1}, \quad 1 \leq m \leq N$$

and obtain the linear Schrodinger problem

$$\mathcal{L}\hat{u}_m = \lambda_m \hat{u}_m, \qquad \mathcal{L} = -\underline{m}\partial_t^2 - k(t),$$

where $\lambda_m = 4K_2 \sin^2\left(\frac{q_m}{2}\right)$.

Bifurcation theory: $c = 0$

Due to Dirichlet conditions, we use the discrete Fourier sine modes:

$$u_n(t) = \sum_{m=1}^{N} \hat{u}_m(t) \sin(q_m n), \quad q_m := \frac{\pi m}{N+1}, \quad 1 \leq m \leq N$$

and obtain the linear Schrodinger problem

$$\mathcal{L} \hat{u}_m = \lambda_m \hat{u}_m, \qquad \mathcal{L} = -\underline{m} \partial_t^2 - k(t),$$

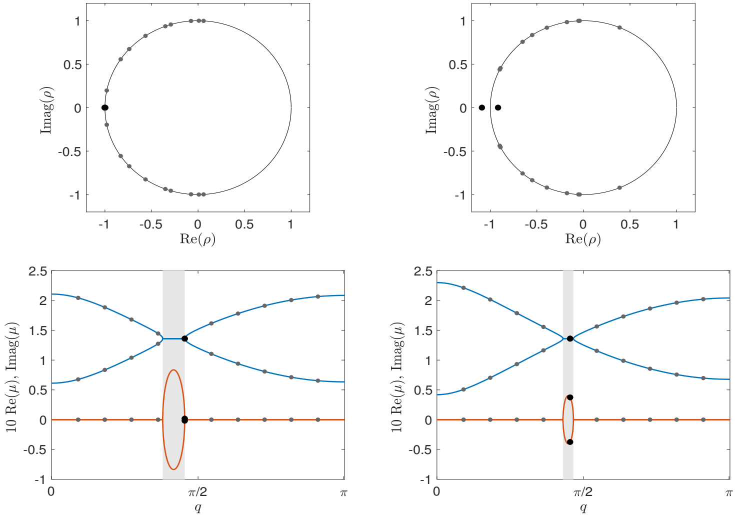
where $\lambda_m = 4K_2 \sin^2\left(\frac{q_m}{2}\right)$.

The spectrum of $\mathcal{L}$ in $L^2(\mathbb{R})$ is purely continuous:

$$\sigma(\mathcal{L}) = [\nu_0, \mu_1] \cup [\mu_2, \nu_1] \cup [\nu_2, \mu_3] \cup [\mu_4, \nu_3] \cup \cdots$$

We are looking for a bifurcation case of $k(t)$ when $\lambda_{m_0} = \mu_1$ or $\lambda_{m_0} = \mu_2$ for one $m_0 \in \{1, 2, \ldots, N\}$ (double-period bifurcation).

Bifurcation theory: $c = 0$



Generalized localized-in-time breathers

Assume three conditions:

- Spectral Assumption: There exists $k_0(t+2\pi) = k_0(t)$ such that all Floquet exponents are purely imaginary, With the exception of two exponents at $\ell_0 = \frac{i}{2}$, all other are simple and nonzero.

$$k(t) = k_0(t) + \delta\varepsilon^2, \quad \delta = \pm 1.$$

- Non-degeneracy: The numerical coefficient χ of the normal form is nonzero and focusing:

$$\frac{1}{2}\lambda_1''(\ell_0)A'' + \delta A + \chi A^3 = 0,$$

with $\delta = -\text{sgn}(\lambda_1''(\ell_0)) = -\text{sgn}(\chi)$, e.g. $K_3 = 0$ and $K_4 \neq 0$.

- Reversibility in time: The time-periodic coefficient satisfies $k(t - t_0) = k(t_0 - t)$ for at least one $t_0 \in [0, 2\pi]$ (two generally).

Generalized localized-in-time breathers

Theorem (C. Chong, D.P., G. Schneider, SIADS 24 (2025) 894)

Under the three conditions, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and every $M \geq 3$ the FPU system possesses two generalized homoclinic solutions $U_{\rm hom}^{\pm}(t) : [-\varepsilon^{-M+2}, \varepsilon^{-M+2}] \to \mathbb{R}^N$ satisfying

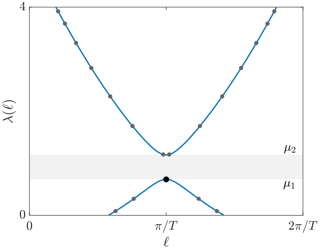
$$\sup_{t \in [-\varepsilon^{-M+2}, \varepsilon^{-M+2}]} \|U_{\rm hom}^{\pm}(t) - \mathcal{U}^{\pm}(t)\| + \|(U_{\rm hom}^{\pm})'(t) - (\mathcal{U}^{\pm})'(t)\| \leq C\varepsilon^{M-1}$$

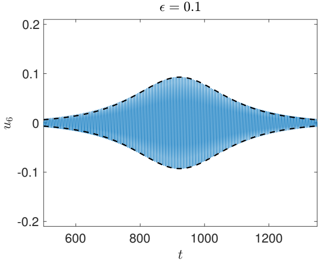
where $\mathcal{U}^{\pm}(t) : \mathbb{R} \to \mathbb{R}^N$ satisfy $\lim_{|t| \to \infty} \|\mathcal{U}^{\pm}(t)\| + \|(\mathcal{U}^{\pm})'(t)\| = 0$ and can be approximated as

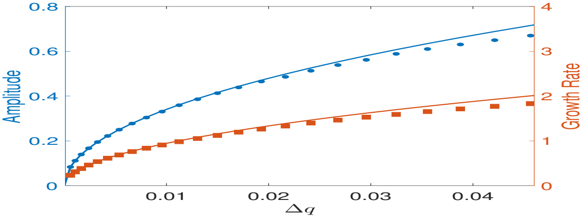
$$(\mathcal{U}^{\pm})_n(t) = \pm \varepsilon A(\varepsilon t) g(t) \sin(q_{m_0} n) + \mathcal{O}(\varepsilon^2),$$

where $g(t + 2\pi) = -g(t)$ and $A(\tau) = \alpha \operatorname{sech}(\beta \tau)$ are uniquely defined with some $\alpha, \beta > 0$.

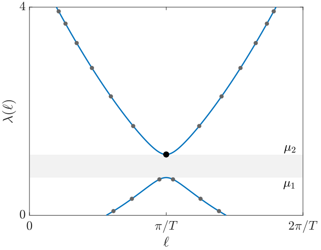
Numerical illustration: bifurcation from $\lambda_{m_0} = \mu_1$, $K_4 < 0$

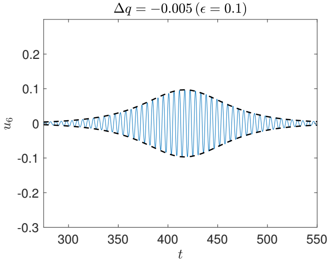


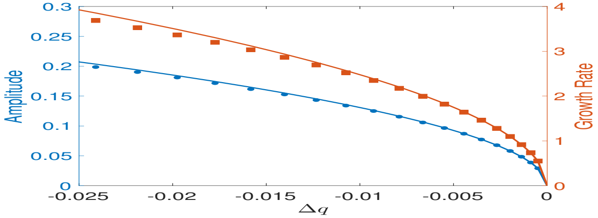




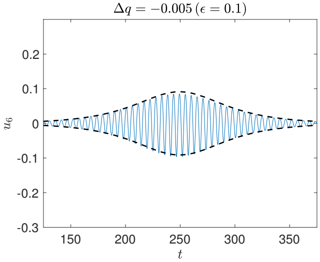
Numerical illustration: bifurcation from $\lambda_{m_0} = \mu_2$, $K_4 > 0$

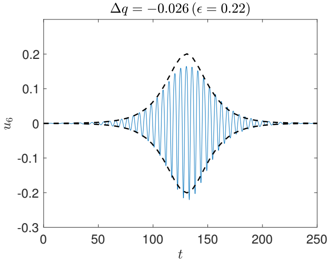


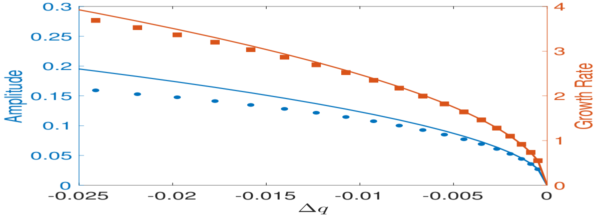




Numerical illustration: bifurcation from $\lambda_{m_0} = \mu_2$, $K_3 \neq 0$







Justification of the homoclinic solutions

Step 1: Bifurcation setup.

Let $k(t) = k_0(t) + \delta\varepsilon^2$ and pick $k_0(t)$ so that $\lambda_{m_0} = \mu_1$ for one $m_0 \in \{1, 2, \dots, N\}$. Assume no other Floquet multipliers at ± 1.

Justification of the homoclinic solutions

Step 1: Bifurcation setup.

Let $k(t) = k_0(t) + \delta\varepsilon^2$ and pick $k_0(t)$ so that $\lambda_{m_0} = \mu_1$ for one $m_0 \in \{1, 2, \dots, N\}$. Assume no other Floquet multipliers at ± 1.

Step 2: Formal derivation of the normal form. Expanding

$$u_n(t) = \varepsilon U_n^{(1)}(t) + \varepsilon^2 U_n^{(2)}(t) + \varepsilon^3 U_n^{(3)}(t) + \mathcal{O}(\varepsilon^4),$$

we select the leading order in the form

$$U_n^{(1)}(t) = A(\varepsilon t) g(t) \sin(q_{m_0} n),$$

where $g(t + 2\pi) = -g(t)$ is the bifurcating mode of $\mathcal{L}_0 g = \mu_1 g$ with $\mu_1 = K_2 \omega^2(q_{m_0})$, $\omega^2(q) := 4\sin^2(\frac{q}{2})$.

Justification of the homoclinic solutions

At the order of $\mathcal{O}(\varepsilon^2)$, we get

$$\mathcal{L}_0 U_n^{(2)} + \Delta U_n^{(2)} = 2\underline{m} A'(\tau) g'(t) \sin(q_{m_0} n) + K_3 A(\tau)^2 g(t)^2 F_n^{(2)},$$

where $\tau = \varepsilon t$ and $F_n^{(2)} = -2\sin(q_{m_0})(1-\cos(q_{m_0}))\sin(2q_{m_0} n)$.

The solution for $U_n^{(2)}(t)$ can be written in the form

$$U_n^{(2)}(t) = A'(\tau) h_1(t) \sin(q_{m_0} n) + \chi_2 A(\tau)^2 h_2(t) \sin(2q_{m_0} n),$$

where

$$\begin{aligned} (\mathcal{L}_0 - K_2\omega^2(q_{m_0}))h_1 &= 2\underline{m} g'(t), \\ (\mathcal{L}_0 - K_2\omega^2(2q_{m_0}))h_2 &= -2\sin(q_{m_0})(1-\cos(q_{m_0}))g(t)^2. \end{aligned}$$

The unique solution for $h_1(t+T) = -h_1(t)$ and $h_2(t+T) = h_2(t)$ exists under the spectral assumption.

Justification of the homoclinic solutions

At the order of $\mathcal{O}(\varepsilon^3)$, we get

$$\begin{aligned}
\mathcal{L}_0 U_n^{(3)} + \Delta U_n^{(3)} &= \delta U_n^{(1)} + 2\underline{m}\partial_\tau \partial_t U_n^{(2)} + \underline{m}\partial_\tau^2 U_n^{(1)} \\
&+ 2K_3 \left[(U_{n+1}^{(1)} - U_n^{(1)})(U_{n+1}^{(2)} - U_n^{(2)}) - (U_n^{(1)} - U_{n-1}^{(1)})(U_n^{(2)} - U_{n-1}^{(2)}) \right] \\
&- K_4 \left[(U_{n+1}^{(1)} - U_n^{(1)})^3 - (U_n^{(1)} - U_{n-1}^{(1)})^3 \right].
\end{aligned}$$

Projection to the resonant mode $g(t)\sin(q_{m_0} n)$ yields the normal form:

$$\frac{1}{2}\lambda_1''(\ell_0)A''(\tau) + \delta A(\tau) + \chi A(\tau)^3 = 0,$$

where $\chi \neq 0$ under the non-degeneracy assumption.

Justification of the homoclinic solutions

Step 3: Justification of the normal form. The normal form theorem near the double period bifurcation (Iooss–Adelmeyer, 1998) after

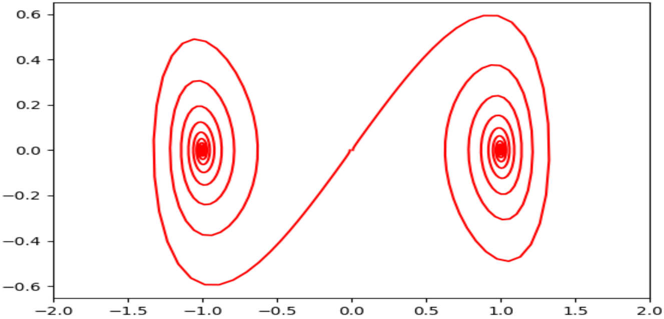
- diagonalization of the linear system in Fourier sine modes,
- near-identity transformations to push remainder terms in the higher order of ε,
- persistence of reversible homoclinic orbits in the 2-dimensional perturbed normal form under the reversibility assumption on $k(t)$
- persistence analysis for the remaining $(2N-2)$-dimensional system by using variation of parameters (Duhamel's formual) and Gronwall-type estimates of the remainder terms.

3. Transition fronts due to dissipation $c > 0$

If $c = \varepsilon \tilde{c} > 0$ is small, the normal form equation is

$$\frac{1}{2}\lambda_1''(\ell_0)\left[A''(\tau) + \frac{\tilde{c}}{\underline{m}}A'(\tau)\right] + \delta A(\tau) + \chi A(\tau)^3 = 0,$$

and the homoclinic orbits transform into the heteroclinic orbits.



3. Transition fronts due to dissipation $c > 0$

Theorem (C. Chong, D.P., G. Schneider, SIADS 24 (2025) 894)

Under the spectral and non-degeneracy assumptions, there exists an $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *the FPU system possesses two anti-periodic solutions* $\mathcal{U}^{\pm}_{\rm antiper}$ *such that*

$$\sup_{t \in \mathbb{R}} \|\mathcal{U}^{\pm}_{\rm antiper}(t)\| + \|(\mathcal{U}^{\pm}_{\rm antiper})'(t)\| \leq \; C_0 \varepsilon$$

and two heteroclinic solutions $U^{\pm}_{\rm het} \in C^1(\mathbb{R}, \mathbb{R}^N)$ *such that*

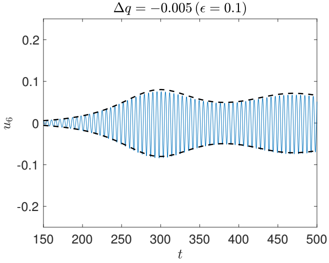
$$\lim_{t \to -\infty} U^{\pm}_{\rm het}(t) = 0, \qquad \lim_{t \to -\infty} (U^{\pm}_{\rm het})'(t) = 0$$

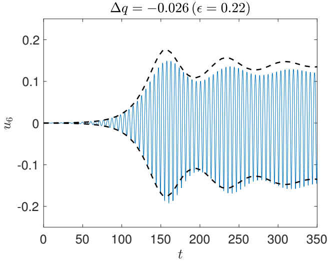
and

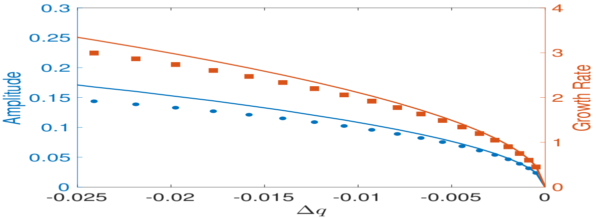
$$\lim_{t \to \infty} \inf_{t_0 \in [0, 2\pi]} \|U^{\pm}_{\rm het}(t) - \mathcal{U}^{\pm}_{\rm antiper}(t + t_0)\| = 0.$$

3. Transition fronts due to dissipation $c > 0$

Numerical illustration with $K_3, K_4 \neq 0$:

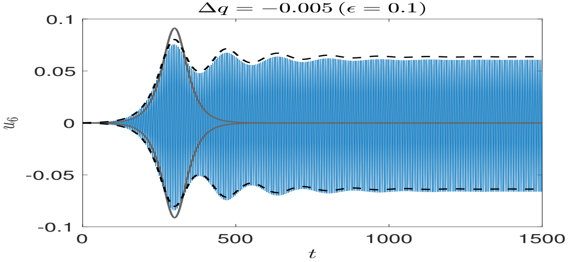


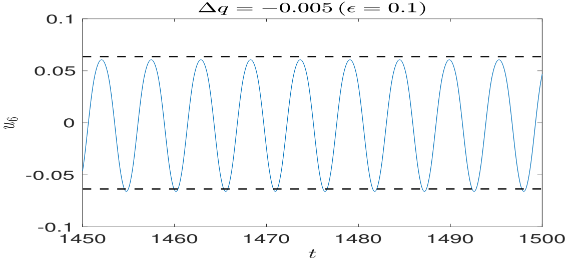




3. Transition fronts due to dissipation $c > 0$

Longer computation:





4. Conclusion

- We considered the generalized breathers (localized-in-time pulses) in the FPU lattice.
- These solutions are recovered in the dynamical systems on a long but finite scale.
- Numerical experiments do not often distinguish between true and generalized breathers.
- In the presence of dissipation, the generalized breathers transform into transition fronts (heteroclinic orbits).

MANY THANKS FOR YOUR ATTENTION!