Moving gap solitons in periodic potentials

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Reference: Mathematical Methods for Physical Sciences, Submitted

Dynamics Days, Loughborough, July 9–13, 2007
Motivations

**Gap solitons** are localized stationary solutions of nonlinear PDEs with space-periodic coefficients which reside in the spectral gaps of associated linear operators.

**Examples:** Complex-valued Maxwell equation

\[ \nabla^2 E - E_{tt} + (V(x) + \sigma |E|^2) E_{tt} = 0 \]

and the Gross–Pitaevskii equation

\[ i E_t = -\nabla^2 E + V(x)E + \sigma |E|^2 E, \]

where \( E(x, t) : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{C}, V(x) = V(x + 2\pi e_j) : \mathbb{R}^N \mapsto \mathbb{R}, \)

and \( \sigma = \pm 1. \)
**Existence of stationary solutions**

Stationary solutions $E(x, t) = U(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfy a nonlinear elliptic problem with a periodic potential

$$\nabla^2 U + \omega U = V(x)U + \sigma|U|^2U$$

**Theorem:**[Pankov, 2005] Let $V(x)$ be a real-valued bounded periodic potential. Let $\omega$ be in a finite gap of the spectrum of $L = -\nabla^2 + V(x)$. There exists a non-trivial weak solution $U(x) \in H^1(\mathbb{R}^N)$, which is (i) real-valued, (ii) continuous on $x \in \mathbb{R}^N$ and (iii) decays exponentially as $|x| \to \infty$.

**Remark:** Additionally, there exists a localized solution $U(x) \in H^1(\mathbb{R}^N)$ in the semi-infinite gap for $\sigma = -1$ (NLS soliton).
Coupled-mode theory for gap solitons

Stationary gap solitons can be approximated asymptotically by the coupled-mode theory in one dimension ($N = 1$) in the limit of small-amplitude potentials: $V(x) = \epsilon (1 - \cos x)$ for small $\epsilon$.

The finite-band spectrum of $L = -\partial_x^2 + V(x)$ is shown here:

Coupled-mode equations are derived with asymptotic multi-scale expansions:

$$E(x, t) = \sqrt{\epsilon} \left[ a(\epsilon x, \epsilon t) e^{\frac{ix}{2}} + b(\epsilon x, \epsilon t) e^{-\frac{ix}{2}} + O(\epsilon) \right] e^{-\frac{it}{4}}.$$
The vector \((a, b) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}^2\) satisfies asymptotically the coupled-mode system:

\[
\begin{align*}
    i(a_T + a_X) + V_2 b &= \sigma(|a|^2 + 2|b|^2) a, \\
    i(b_T - b_X) + V_{-2} a &= \sigma(2|a|^2 + |b|^2) b,
\end{align*}
\]

where \(X = \epsilon x, T = \epsilon t\), and \(V_2 = \bar{V}_{-2}\) are Fourier coefficients of \(V(x)\). Stationary gap solitons are obtained in the analytic form

\[a(X, T) = a(X) e^{-i\Omega T}, \quad b(X, T) = b(X) e^{-i\Omega T},\]

\[a(X) = \bar{b}(X) = \frac{\sqrt{2}}{\sqrt{3}} \frac{\sqrt{|V_2|^2 - \Omega^2}}{\sqrt{|V_2| - \Omega \cosh(\kappa X) + i \sqrt{|V_2| + \Omega \sinh(\kappa X)}}},\]

where \(\kappa = \sqrt{|V_2|^2 - \Omega^2}\) and \(|\Omega| < |V_2|\).
Moving gap solitons

Moving gap solitons are obtained in the analytic form

\[ a = \left( \frac{1 + c}{1 - c} \right)^{1/4} A(\xi) e^{-i \mu \tau}, \quad b = \left( \frac{1 - c}{1 + c} \right)^{1/4} B(\xi) e^{-i \mu \tau}, \quad |c| < 1, \]

where

\[ \xi = \frac{X - cT}{\sqrt{1 - c^2}}, \quad \tau = \frac{T - cX}{\sqrt{1 - c^2}} \]

and, since \( |A|^2 - |B|^2 \) is constant in \( \xi \in \mathbb{R} \), then

\[ A = \phi(\xi) e^{i \varphi(\xi)}, \quad B = \bar{\phi}(\xi) e^{i \varphi(\xi)}, \]

with \( \phi \) and \( \varphi \) being solutions of the system

\[ \varphi' = \frac{-2c \sigma |\phi|^2}{(1 - c^2)}, \quad i \phi' = V_2 \bar{\phi} - \mu \phi + \sigma \frac{(3 - c^2)}{(1 - c^2)} |\phi|^2 \phi. \]
Main Questions: (a) Can we justify the use of the coupled-mode
theory to approximate stationary gap solitons?
   YES: D.P., G.Schneider, Asymptotic Analysis (2007)
(b) Can we justify the use of the coupled-mode theory to
approximate moving gap solitons?
   NO: this work

Theorem: [Goodman, Weinstein, Holmes, 2001; Schneider, Uecker,
2001:] Let \((a, b) \in C([0, T_0], H^3(\mathbb{R}, C^2))\) be solutions of the
time-dependent coupled-mode system for a fixed \(T_0 > 0\). There
exists \(\epsilon_0, C > 0\) such that for all \(\epsilon \in (0, \epsilon_0)\) the Gross–Pitaevskii
equation has a local solution \(E(x, t)\) and

\[
\|E(x, t) - \sqrt{\epsilon} \left[ a(\epsilon x, \epsilon t) e^{i(kx-\omega t)} + b(\epsilon x, \epsilon t) e^{i(-kx-\omega t)} \right] \|_{H^1(\mathbb{R})} \leq C\epsilon
\]

for some \((k, \omega)\) and any \(t \in [0, T_0/\epsilon]\).
Assumptions of the main theorem

**Assumption:** Let $V(x)$ be a smooth $2\pi$-periodic real-valued function with zero mean and symmetry $V(x) = V(-x)$ on $x \in \mathbb{R}$, such that

$$V(x) = \sum_{m \in \mathbb{Z}} V_{2m} e^{imx} : \sum_{m \in \mathbb{Z}} (1 + m^2)^s |V_{2m}|^2 < \infty,$$

for some $s \geq 0$, where $V_0 = 0$ and $V_{2m} = V_{-2m} = \bar{V}_{-2m}$.

**Definition:** The moving gap soliton of the coupled-mode system is said to be a reversible homoclinic orbit if $(A, B)$ decays to zero at infinity and $A(\xi) = \bar{A}(-\xi)$, $B(\xi) = \bar{B}(-\xi)$ in the parametrization above.

**Remark:** If $V(x) = V(-x)$ and $U(x)$ is a solution of

$$\nabla^2 U + \omega U = V(x)U + \sigma |U|^2 U,$$

then $\bar{U}(-x)$ is also a solution.
**Main Theorem**

**Theorem:** Let $V(x)$ satisfy the assumption and $V_{2n} \neq 0$ for a $n \in \mathbb{N}$.

Let $\omega = \frac{n^2}{4} + \epsilon \Omega$ with $|\Omega| < \Omega_0 = |V_{2n}| \frac{\sqrt{n^2-c^2}}{n}$.

Let $0 < c < n$, such that $\frac{n^2+c^2}{2c} \notin \mathbb{Z}$. Fix $N \in \mathbb{N}$.

Then, there exists $\epsilon_0, L, C > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the Gross–Pitaevskii equation has a solution in the form

$$E(x, t) = e^{-i\omega t} \psi(x, y),$$

where $y = x - ct$ and the function $\psi(x, y)$ is a periodic (anti-periodic) function of $x$ for even (odd) $n$,

satisfying the reversibility constraint $\psi(x, y) = \bar{\psi}(x, -y)$, and

$$\left| \psi(x, y) - \epsilon^{1/2} \left( a_\epsilon(\epsilon y) e^{\frac{inx}{2}} + b_\epsilon(\epsilon y) e^{-\frac{inx}{2}} \right) \right| \leq C_0 \epsilon^{N+1/2},$$

for all $x \in \mathbb{R}$ and $y \in [ -L/\epsilon^{N+1}, L/\epsilon^{N+1} ]$. Here

$a_\epsilon(Y) = a(Y) + O(\epsilon)$ on $Y = \epsilon y \in \mathbb{R}$ is an exponentially decaying reversible solution, while $a(Y)$ is a solution of the coupled-mode system with $Y = X - cT$. 
Remarks on the Main Theorem

1. The solution $\psi(x, y)$ is a bounded non-decaying function on a large finite interval

$$y \in \left[-L/\epsilon^{N+1}, L/\epsilon^{N+1}\right] \subset \mathbb{R}$$

and we do not claim that the solution $\psi(x, y)$ can be extended to a global bounded function on $y \in \mathbb{R}$.

2. Since the homoclinic orbit $(a, b)$ of the coupled-mode system is single-humped, the traveling solution $\psi(x, y)$ is represented by a single bump surrounded by bounded oscillatory tails.

3. The solution $(a_\epsilon, b_\epsilon)$ is defined up to the terms of $O(\epsilon^N)$ and it satisfies an extended coupled-mode system, which is a perturbation of the coupled-mode system with $Y = X - cT$. 


Spatial dynamics formulation

Set $E(x, t) = e^{-i\omega t}\psi(x, y)$ with $y = x - ct$ and a parameter $\omega$. For traveling solutions, $c \neq 0$ and we set $c > 0$. Then,

$$(\omega - ic\partial_y + \partial_x^2 + 2\partial_x\partial_y + \partial_y^2) \psi = V(x)\psi + \sigma |\psi|^2\psi.$$ 

We consider functions $\psi(x, y)$ being $2\pi$-periodic or $2\pi$-antiperiodic in $x$ and bounded in $y$. Therefore,

$$\psi(x, y) = \sum_{m \in \mathbb{Z}'} \psi_m(y)e^{imx},$$

such that $\psi_m(y)$ satisfy the nonlinear system of coupled ODEs:

$$\psi_m'' + i(m - c)\psi_m' + \left(\omega - \frac{m^2}{4}\right)\psi_m = \sum_{m_1 \in \mathbb{Z}'} V_{m-m_1}\psi_{m_1} + \text{N.T.}$$
Eigenvalues of the spatial dynamics

Linearization of the system with \( \psi_m(y) = e^{\kappa y} \delta_{m,m_0} \) gives roots \( \kappa = \kappa_m \) in the quadratic equation with \( \omega = \frac{n^2}{4} \):

\[
\kappa^2 + i(m - c)\kappa + \omega - \frac{m^2}{4} = 0, \quad \forall m \in \mathbb{Z}'.
\]

- For \( m > m_0 = \left\lfloor \frac{n^2+c^2}{2c} \right\rfloor \), all roots are complex-valued.
- For \( m \leq m_0 \), all roots are purely imaginary. The zero root is semi-simple of multiplicity two. All other roots are semi-simple of maximal multiplicity three.
- If \( c \) is irrational, all non-zero roots are simple but may approach to each other arbitrarily closer.
Hamiltonian formulation

Let $\phi_m(y) = \psi'_m(y) - \frac{i}{2}(c - m)\psi_m(y)$ and rewrite the system of ODEs:

\[
\begin{align*}
\frac{d\psi_m}{dy} &= \phi_m + \frac{i}{2}(c - m)\psi_m \\
\frac{d\phi_m}{dy} &= -\frac{1}{4}(n^2 + c^2 - 2cm)\psi_m + \frac{i}{2}(c - m)\phi_m - \epsilon\Omega\psi_m + N.T.
\end{align*}
\]

The system is Hamiltonian in canonical variables $(\psi, \phi, \bar{\psi}, \bar{\phi})$. The vector field maps a domain in $D$ to a range in $X$, where

\[
D = \left\{ (\psi, \phi, \bar{\psi}, \bar{\phi}) \in l_{s+1}^2(\mathbb{Z}, \mathbb{C}^4) \right\}, \quad X = \left\{ (\psi, \phi, \bar{\psi}, \bar{\phi}) \in l_s^2(\mathbb{Z}, \mathbb{C}^4) \right\}
\]

and $l_s^2(\mathbb{Z})$ is a Banach algebra for any $s > \frac{1}{2}$. The phase space is $X$. 
Symmetries

Solutions are invariant under the reversibility transformation

\[ \psi(y) \mapsto \overline{\psi}(-y), \quad \phi(y) \mapsto -\overline{\phi}(-y), \quad \forall y \in \mathbb{R}. \]

and the gauge transformation

\[ \psi(y) \mapsto e^{i\alpha} \psi(y), \quad \phi(y) \mapsto e^{i\alpha} \phi(y), \quad \forall \alpha \in \mathbb{R}. \]

Reversible solutions satisfy the constraints:

\[ \psi(-y) = \overline{\psi}(y), \quad \phi(-y) = -\overline{\phi}(y), \quad \forall y \in \mathbb{R}, \]

which means that the trajectory intersects the reversibility surface

\[ \Sigma_r = \{(\psi, \phi, \overline{\psi}, \overline{\phi}) \in D : \quad \text{Im}\psi = 0, \quad \text{Re}\phi = 0\}. \]
Let $Z_- = \{m \in \mathbb{Z}' : m \leq m_0\}$, $Z_+ = \{m \in \mathbb{Z}' : m > m_0\}$ and

\[ Z_- : \psi_m = \frac{c_m^+ + c_m^-}{\sqrt[4]{n^2 + c^2 - 2cm}}, \quad \phi_m = \frac{i}{2} \frac{\sqrt[4]{n^2 + c^2 - 2cm}(c_m^+ - c_m^-)}{c_m^+ + c_m^-}, \]

\[ Z_+ : \psi_m = \frac{c_m^+ + c_m^-}{\sqrt[4]{2cm - n^2 - c^2}}, \quad \phi_m = \frac{1}{2} \frac{\sqrt[4]{2cm - n^2 - c^2}(c_m^+ - c_m^-)}{c_m^+ + c_m^-}. \]

The new Hamiltonian system is rewritten in new canonical variables

\[ \forall m \in Z_- : \quad \frac{dc_m^+}{dy} = i \frac{\partial H}{\partial c_m^+}, \quad \frac{dc_m^-}{dy} = -i \frac{\partial H}{\partial c_m^-}, \]

\[ \forall m \in Z_+ : \quad \frac{dc_m^+}{dy} = - \frac{\partial H}{\partial c_m^-}, \quad \frac{dc_m^-}{dy} = \frac{\partial H}{\partial c_m^+}, \]

where $H$ is a new Hamiltonian functions in variables $c^+$ and $c^-$. 
Truncated coupled-mode system

The new Hamiltonian function is

\[ H = \sum_{m \in \mathbb{Z}} (k_m^+|c_m^+|^2 - k_m^-|c_m^-|^2) + \sum_{m \in \mathbb{Z}_+} (\kappa_m^-c_m^-c_m^+ - \kappa_m^+c_m^+c_m^-) + \text{N.T.} \]

Consider the subspace

\[ S = \{ c_m^+ = 0, \; \forall m \in \mathbb{Z}\setminus\{n\}, \quad c_m^- = 0, \; \forall m \in \mathbb{Z}\setminus\{-n\} \} \]

and truncate \( H \) on the subspace \( S \):

\[ H|_S = \epsilon \left[ \frac{\Omega|c_n^+|^2}{n-c} + \frac{\Omega|c_{-n}^-|^2}{n+c} - \frac{V_{2n}(c_n^+c_{-n}^- + c_{-n}^-c_n^+)}{\sqrt{n^2 - c^2}} + \text{N.T.} \right]. \]

The Hamiltonian system for \( (c_n^+, c_{-n}^-) \) is nothing but the coupled-mode system for \( a = \frac{c_n^+}{\sqrt{n-c}} \) and \( b = \frac{c_{-n}^-}{\sqrt{n+c}} \) in \( Y = \epsilon y \).
Extended coupled-mode system

How to avoid formal truncation and to separate the coupled-mode system from the remainder? Use near-identity canonical transformations to obtain the new Hamiltonian function in the form

\[ H = \sum_{m \in \mathbb{Z}_-} \left( k_m^+ |c_m^+|^2 - k_m^- |c_m^-|^2 \right) + \sum_{m \in \mathbb{Z}_+} \left( \kappa_m^- c_m^- \bar{c}_m^+ - \kappa_m^+ c_m^+ \bar{c}_m^- \right) + \epsilon H_S(c_n^+, c_{-n}^-) + \epsilon H_T(c_n^+, c_{-n}^-, c^+, c^-) + \epsilon^{N+1} H_R(c_n^+, c_{-n}^-, c^+, c^-). \]

If \( H_R \equiv 0 \), the subspace \( S \) is invariant subspace of the Hamiltonian system and dynamics on \( S \) is given by a four-dimensional ODE system

\[ \frac{d c_n^+}{dY} = i \frac{\partial H_S}{\partial \bar{c}_n^+}, \quad \frac{d c_{-n}^-}{dY} = -i \frac{\partial H_S}{\partial c_{-n}^-}, \]

where \( Y = \epsilon y \).
Persistence results

**Lemma:** There exists a reversible homoclinic orbit of the extended coupled-mode system which satisfies

\[ |c_n^+(y)| \leq C_+ e^{-\epsilon \gamma |y|}, \quad |c_n^-(y)| \leq C_- e^{-\epsilon \gamma |y|}, \quad \forall y \in \mathbb{R}, \]

for some \( \gamma, C_+, C_- > 0 \) and sufficiently small \( \epsilon \).

**Lemma:** The linearized system at the zero solution is topologically equivalent for sufficiently small \( \epsilon \), except that the double zero eigenvalue at \( \epsilon = 0 \) split into a pair of complex eigenvalues to the left and right half-planes for \( \epsilon > 0 \).

Divide the phase space near the zero solution into

\[ X = X_h \oplus X_c \oplus X_u \oplus X_s \]

and rewrite the system for \( c_0 + c_h \in X_h \) and \( c \in X_c \oplus X_u \oplus X_s \).
Final system of equations

The system of equations

\[
\frac{d\mathbf{c}_h}{dy} = \epsilon \Lambda_h(\mathbf{c}_0)\mathbf{c}_h + \epsilon \mathbf{G}_T(\mathbf{c}_0)(\mathbf{c}_h, \mathbf{c}) + \epsilon^{N+1} \mathbf{G}_R(\mathbf{c}_0 + \mathbf{c}_h, \mathbf{c}),
\]

\[
\frac{d\mathbf{c}}{dy} = \Lambda_c \mathbf{c} + \epsilon \mathbf{F}_T(\mathbf{c}_0 + \mathbf{c}_h, \mathbf{c}) + \epsilon^{N+1} \mathbf{F}_R(\mathbf{c}_0 + \mathbf{c}_h, \mathbf{c}),
\]

where the linearization operator \( \Lambda_h(\mathbf{c}_0) \) has a two-dimensional kernel spanned by \( \mathbf{c}'_0(y) \) and \( \sigma_1 \mathbf{c}_0(y) \) and the remainder terms satisfy the bounds

\[
\| \mathbf{G}_R \|_{X_h} \leq N_R \left( \| \mathbf{c}_0 + \mathbf{c}_h \|_{X_h} + \| \mathbf{c} \|_{X_h^\perp} \right),
\]

\[
\| \mathbf{G}_T \|_{X_h} \leq N_T \left( \| \mathbf{c}_h \|_{X_h}^2 + \| \mathbf{c} \|_{X_h^\perp}^2 \right),
\]

\[
\| \mathbf{F}_T \|_{X'} \leq M_T \left( \| \mathbf{c}_0 + \mathbf{c}_h \|_{X_h} + \| \mathbf{c} \|_{X_h^\perp} \right) \| \mathbf{c} \|_{X_h^\perp}.
\]
Local center-stable manifold

**Theorem:** Let \( a \in X_c, b \in X_s \) and \( (\alpha_1, \alpha_2) \in \mathbb{C}^2 \) be small:

\[
\|a\|_{X_c} \leq C_a \epsilon^N, \quad \|b\|_{X_s} \leq C_b \epsilon^N, \quad |\alpha_1| + |\alpha_2| \leq C_\alpha \epsilon^N.
\]

There exists a family of local solutions \( c_h = c_h(y; a, b, \alpha_1, \alpha_2) \) and \( c = c(y; a, b, \alpha_1, \alpha_2) \) such that

\[
c_c(0) = a, \quad c_s = e^{y \Lambda_s} b + \tilde{c}_s(y), \quad c_h = \alpha_1 s_1(y) + \alpha_2 s_2(y) + \tilde{c}_h(y),
\]

where \( \tilde{c}_s(y) \) and \( \tilde{c}_h(y) \) are uniquely defined and the family of local solutions satisfies the bound

\[
\sup_{\forall y \in [0,L/\epsilon^{N+1}]} \|c_h(y)\|_{X_h} \leq C_h \epsilon^N, \quad \sup_{\forall y \in [0,L/\epsilon^{N+1}]} \|c(y)\|_{X_h^\perp} \leq C \epsilon^N,
\]

for some constants \( C_h, C > 0 \).
Ideas of the proof

1. Use the cut-off function on \( y \in [0, y_0] \) and use the Implicit Function Theorem for components \( c_s, c_u \) resulting in

\[
\|c_{s,u}\|_{C^0_b} \leq C\|F_{s,u}\|_{C^0_b}.
\]

2. Use the cut-off functions on \( y \in [0, y_0] \) and the reversible continuation of solutions on \( y \in [-y_0, 0] \). Then, use the Implicit Function Theorem for component \( c_h \) resulting in

\[
\|c_h\|_{C^0_b} \leq \frac{C'}{\epsilon}\|F_h\|_{C^0_b}.
\]

3. Use variation of constant formula and the Gronwall inequality for component \( c_c \). The bounds are consistent for \( y_0 = L/\epsilon^{N+1} \).
Proof of the main theorem

The local center-stable manifold is extended to a local solution on \( y \in [-y_0, y_0] \) if it intersects the reversibility surface \( \Sigma_r \).

Since \( c_c(0) = a \) is arbitrary, we can set immediately

\[
\text{Im}(a)_m^+ = 0, \quad \forall m \in \mathbb{Z} \setminus \{n\}, \quad \text{Im}(a)_m^- = 0, \quad \forall m \in \mathbb{Z} \setminus \{-n\}.
\]

The other parameters \( b \) and \( (\alpha_1, \alpha_2) \) are not however the initial conditions. They satisfy the set of reversibility constraints

\[
\text{Re}b_m + \text{Re}(\tilde{c}_s)_m(0) = \text{Re}(c_u)_m(0), \quad \text{Im}b_m + \text{Im}(\tilde{c}_s)_m(0) = -\text{Im}(c_u)_m(0)
\]

and

\[
\text{Im}c_n^+(0) = 0, \quad \text{Im}c_n^-(0) = 0.
\]

The first set is solved by the Implicit Function Theorem. The second set is satisfied if \( \alpha_1 = \alpha_2 = 0 \), since the kernel does not satisfy the reversibility but the inhomogeneous solution for \( c_h \) does.
Extensions

We have checked that modified Gross–Pitaevskii equations still possess infinitely many eigenvalues on the imaginary axis:

\[
\begin{align*}
E_{tt} &= E_{xx} + V(x)E + \sigma |E|^2 E, \\
iE_t &= -E_{xx} + iE_{xxt} + V(x)E + \sigma |E|^2 E, \\
iE_n &= -E_{n+1} - E_{n-1} + V_n E_n + \sigma |E_n|^2 E_n.
\end{align*}
\]

In all these equations, there is no hope to construct true homoclinic solution (moving gap soliton) but one can construct a local reversible center-stable manifold, which resembles a single bump surrounded by oscillatory tails.

It is an open problem how to extend this local solution to a global solution defined on the entire line.