Existence of breathers (modulating pulses) in periodic systems via spatial dynamics

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Section 1

Workshop in honor of Michael Plum

- First meeting: July 2016 in LMS Durham Symposium
- Extended visit: KIT, January-July 2022 (Humboldt Award)
- Shorter meetings in 2023

- A. Contreras, D.E. Pelinovsky, and M. Plum, Orbital stability of domain walls in coupled Gross-Pitaevskii systems, SIAM J. Math. Anal. **50** (2018) 810–833
- D.E. Pelinovsky and M. Plum, “Dynamics of black solitons in a regularized nonlinear Schrodinger equation", Proceeding AMS **152** (2024) 1217–1231
- D.E. Pelinovsky and M. Plum, “Stability of black solitons in optical systems with intensity-dependent dispersion", SIAM J. Math. Anal. (2024) in print

Section 2

Breathers and Modulating pulses

Examples of a breather

The standard example is the breather of the sine–Gordon equation:

$$u_{tt} - u_{xx} + \sin(u) = 0,$$

given by the exact solution

$$u(x,t) = 4 \arctan \frac{\sqrt{1-\omega^2}\cos(\omega t)}{\omega \cosh(\sqrt{1-\omega^2}x)}, \quad 0 < \omega < 1.$$

This is the standing breather which also generates a family of moving breathers by the Lorentz transformation:

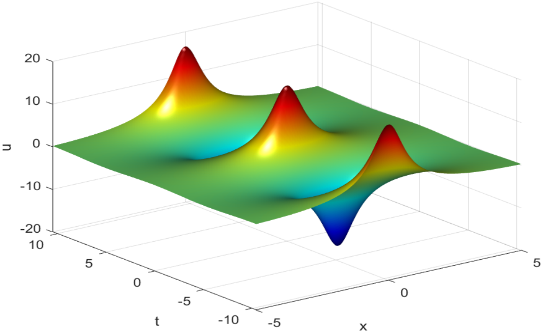
$$u(x,t) = \tilde{u}\left(\frac{x-ct}{\sqrt{1-c^2}}, \frac{t-cx}{\sqrt{1-c^2}}\right), \quad -1 < c < 1.$$

Examples of a breather

The breather solution satisfies

$$u(x, t + T) = u(x, t) \quad \text{and} \quad \lim_{|x| \to \infty} u(x, t) = 0,$$

with $T = 2\pi/\omega$.



Examples of a breather

One striking asymptotic limit is the small-amplitude, slow-scale approximation:

$$u(x,t) = 4\arctan\frac{\sqrt{1-\omega^2}\cos(\omega t)}{\omega\cosh(\sqrt{1-\omega^2}x)}, \quad \omega \in (0,1).$$

If $\varepsilon := \sqrt{1-\omega^2}$ is small, then the power expansions yields

$$u(x,t) = 4\varepsilon \text{sech}(\varepsilon x)\cos(\omega(\varepsilon)t) + \mathcal{O}(\varepsilon^3),$$

with

$$\omega(\varepsilon) = \sqrt{1-\varepsilon^2} = 1 - \frac{1}{2}\varepsilon^2 + \mathcal{O}(\varepsilon^4).$$

Small-amplitude expansions

This suggest the reduction of the sine–Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0,$$

with the small-amplitude, slow-scale expansions

$$u(x,t) = \varepsilon[A(\varepsilon x, \varepsilon^2 t)e^{it} + \bar{A}(\varepsilon x, \varepsilon^2 t)e^{-it}] + \mathcal{O}(\varepsilon^3).$$

Since $\sin(u) = u - \frac{1}{6}u^3 + \mathcal{O}(u^5)$ and $e^{\pm it}$ are in the null space of $1 + \partial_t^2$ in $L^2_{\rm per}$, we get the NLS equation for $A = A(\xi, \tau)$ from the solvability condition in $L^2_{\rm per}$ at the order of $\mathcal{O}(\varepsilon^3)$:

$$2iA_\tau - A_{\xi\xi} - \frac{1}{2}|A|^2 A = 0.$$

The breather corresponds to the NLS soliton $A(\xi, \tau) = 2\mathrm{sech}(\xi)e^{-\frac{i}{2}\tau}$.

Small-amplitude expansions

However, the expansions fail for non-integrable versions of the wave equation, e.g. for the ϕ^4 theory:

$$u_{tt} - u_{xx} + u - \frac{1}{6}u^3 = 0.$$

- H. Segur, M. D. Kruskal, Phys. Rev. Lett. 58 (1987), 747
- J. Denzler, Commun. Math. Phys. 158 (1993) 397
- B. Birnir, H.P. McKean, A. Weinstein, CPAM 47 (1994) 1043
- Justification of the NLS approximation holds only on long but finite time intervals:

$$\sup_{t \in [0, \tau_0 \varepsilon^{-2}]} |u(\cdot, t) - \varepsilon A(\varepsilon \cdot, \varepsilon^2 t) e^{it} - \varepsilon \bar{A}(\varepsilon \cdot, \varepsilon^2 t) e^{-it}\|_{L^\infty} \leq C \varepsilon^3.$$

Small-amplitude expansions

The breather solutions can be thought to be a solution of the form

$$u(x,t) = v(\xi,\theta), \quad \xi := x - ct, \quad \theta := kx - \omega t$$

for some approrpriately choosen parameters c, k, ω and with boundary conditions

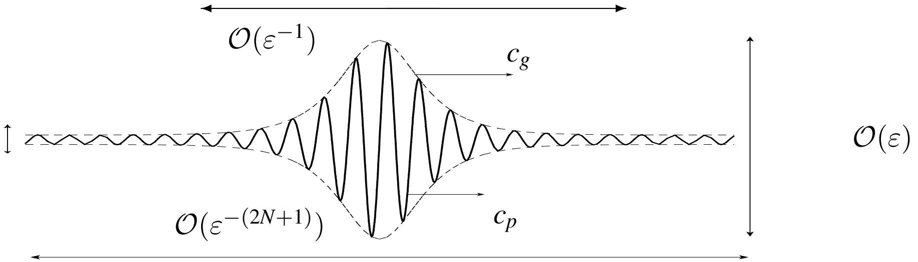
$$u(x,\theta+2\pi) = u(x,\theta) \quad \text{and} \quad \lim_{|\xi|\to\infty} v(\xi,\theta) = 0.$$

The PDE is converted to the spatial dynamical system in ξ by using Fourier series in θ. A center manifold does not allow us generally to construct a homoclinic orbit with zero boundary conditions.

- M. Groves and G. Schneider, Comm. Math. Phys. 219 (2001); J. Diff. Eqs. 219 (2005); Comm. Math. Phys. 278 (2008).

Small-amplitude expansions

Instead of breathers, we would then have modulating pulses which are not trully localized (also called generalized breathers).



Breathers versus modulating pulses

Besides integrable systems, true breathers exist in some models:

- Lattices with weak coupling:

$$\ddot{u}_n - \epsilon^2 (\Delta u)_n + u_n + u_n^3 = 0, \qquad n \in \mathbb{Z}.$$

  S. Aubry & R. MacKay (1994); D.P., T. Penati, S. Paleari (2020)

- Systems with periodic coefficients

$$s(x) u_{tt} - u_{xx} - \rho(x) u + u^3 = 0, \quad s(x+2\pi) = s(x), \; \rho(x+2\pi) = \rho(x).$$

  C. Blank, M. Chirilus, V. Lescarret, G. Schneider (2011);
  A. Hirsch & W. Reichel (2019); S. Kohler & W. Reichel (2022)

- Curl–curl wave equations: M. Plum & W. Reichel (2016), (2023)

Breathers versus modulating pulses

In more general models, modulating pulses exist instead of breathers:

- Standing modulating pulse solutions of the wave equation with periodic coefficients

$$u_{tt} - u_{xx} - \rho(x)u + u^3 = 0, \quad \rho(x + 2\pi) = \rho(x).$$

V. Lescarret, G. Schneider (2009); T. Dohnal, D. Rudolf (2020)

- Traveling modulating pulse solutions of the Gross-Pitaevskii equation with periodic potentials:

$$i\psi_t = -\psi_{xx} + \rho(x)\psi + |\psi|^2\psi, \quad \rho(x + 2\pi) = \rho(x)$$

D.P & G. Schneider (2008); D.P. (2011);

Breathers versus modulating pulses

No results for traveling modulating pulse solutions in the wave equation with periodic coefficients so far.

$$u_{tt} - u_{xx} + \rho(x)u = \gamma u^3, \quad \rho(x + 2\pi) = \rho(x).$$

Here traveling modulating pulses have three spatial scales:

$$\xi = x - ct, \quad \theta = kx - \omega t, \quad x.$$

T. Dohnal, D.P., G. Schneider, Nonlinearity (2024) under review.

Section 3

Traveling modulating pulses in the wave equation with periodic coefficients

Linear theory and traveling modulating pulses

Consider the linear wave equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) + \rho(x) u(x,t) = 0, \quad \rho(x + 2\pi) = \rho(x),$$

with 2π-periodic, bounded, and positive coefficient ρ.

Linear theory and traveling modulating pulses

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with 2π-periodic, bounded, and positive coefficient ρ.

Solutions are given by the family of Bloch modes:

$$u(x,t) = e^{\pm i\omega_n(l)t} e^{ilx} f_n(l,x), \quad n \in \mathbb{N}, \quad l \in \mathbb{B} := \mathbb{R} \backslash \mathbb{Z},$$

where $f_n(l,x) = f_n(l,x+2\pi)$ and $f_n(l,x) = f_n(l+1,x)e^{\mathrm{i}x}$ are $L^2([0,2\pi])$ normalized eigenfunctions and

$$0 < \omega_1(l) \leq \omega_2(l) \leq \cdots \leq \omega_n(l) \leq \omega_{n+1}(l) \leq \dots \quad \forall l \in \mathbb{B},$$

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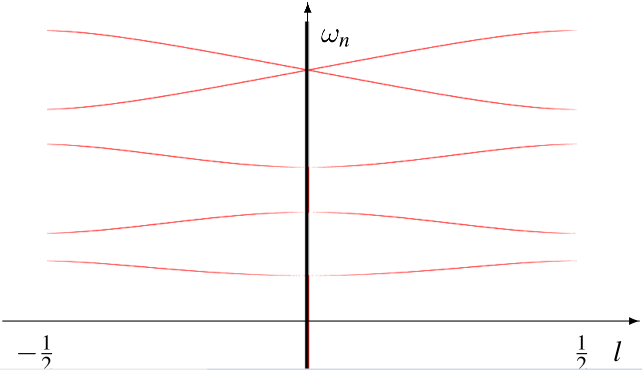

Linear theory and traveling modulating pulses

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with 2π-periodic, bounded, and positive coefficient ρ.

For fixed $n_0 \in \mathbb{N}$ and $l_0 \in \mathbb{B}$, we can approximate the traveling modulating pulse by

$$u_{\mathrm{app}}(x,t) = \varepsilon A(\varepsilon(x - c_g t), \varepsilon^2 t) f_{n_0}(l_0, x) \mathrm{e}^{\mathrm{i} l_0 x} \mathrm{e}^{-\mathrm{i}\omega_{n_0}(l_0)t} + c.c.,$$

where $c_g = \omega'_{n_0}(l_0)$, and $A = A(X,T)$ is a soliton of the NLS equation:

$$2\mathrm{i}\partial_T A + \omega''_{n_0}(l_0)\partial_X^2 A + \gamma_{n_0}(l_0)|A|^2 A = 0,$$

with $\gamma_{n_0}(l_0) = 3\|f_{n_0}(\ell_0,\cdot)\|_{L^4}^4 / \omega_{n_0}(l_0)$.

Main theorem [T. Dohnal, D.P., G. Schneider (2024)]

Choose $n_0 \in \mathbb{N}$ and $l_0 \in \mathbb{B}$ such that $\omega_n(l_0) \neq \omega_{n_0}(l_0), \forall n \neq n_0$, $\omega'_{n_0}(l_0) \neq \pm 1$, $\omega''_{n_0}(l_0) \neq 0$, and

$$\omega_n^2(ml_0) \neq m^2 \omega_{n_0}^2(l_0), \quad m \in \{3, 5, \dots 2N+1\}, \quad \forall n \in \mathbb{N}.$$

There are $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exist traveling modulating pulse solutions of the semi-linear wave equation:

$$u(x,t) = v(\xi, z, x) \quad \text{with } \; \xi = x - c_g t, \;\; z = l_0 x - \omega t,$$

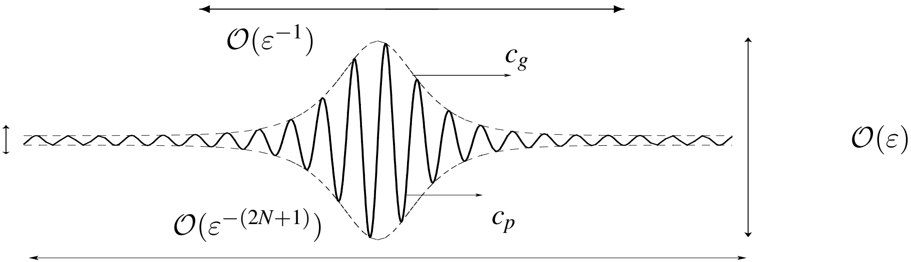
with $v \in C^2([-\varepsilon^{-(2N+1)}, \varepsilon^{-(2N+1)}], \mathcal{X})$ satisfying

$$\sup_{\xi \in [-\varepsilon^{-(2N+1)}, \, \varepsilon^{-(2N+1)}]} |v(\xi, z, x) - h(\xi, z, x)| \leq C \varepsilon^{2N},$$

where $\mathcal{X} := H^2_{\rm per}(\mathbb{T}, L^2(\mathbb{T})) \cap H^1_{\rm per}(\mathbb{T}, H^1_{\rm per}(\mathbb{T})) \cap L^2(\mathbb{T}, H^2_{\rm per}(\mathbb{T}))$.
The function $h \in C^2(\mathbb{R}, \mathcal{X})$ satisfies

$$\lim_{|\xi| \to \infty} h(\xi, z, x) = 0 \quad \text{and} \quad \sup_{\xi, z, x \in \mathbb{R}} \left| h(\xi, z, x) - u_{\rm app}(\xi, z, x) \right| \leq C \varepsilon^2.$$

The illustration of the main result is the same picture:



Some remarks about the main result

As a consequence, the modulating pulses are relevant for the initial-value problem for the wave equation.

Theorem

Let v *be the constructed solution and take an arbitrary function* $\phi \in C^2(\mathbb{R} \setminus [-\varepsilon^{-(2N+1)}, \varepsilon^{-(2N+1)}], \mathcal{X})$ *such that*

$$v_{\text{ext}}(\xi, z, x) := \begin{cases} v(\xi, x, z), & (\xi, x, z) \in [-\varepsilon^{-(2N+1)}, \varepsilon^{-(2N+1)}] \times \mathbb{R} \times \mathbb{R}, \\ \phi(\xi, x, z), & (\xi, x, z) \in \text{otherwise} \end{cases}$$

satisfies $v_{\text{ext}} \in C^2(\mathbb{R}, \mathcal{X})$. *Let* $u_0(x) := v_{\text{ext}}(x, \ell_0 x, x)$ *and*

$$u_1(x) := -c_g \partial_\xi v_{\text{ext}}(x, \ell_0 x, x) - \omega \partial_z v_{\text{ext}}(x, \ell_0 x, x).$$

The corresponding solution of the wave equation satisfies $u(x,t) = v(x - c_g t, l_0 x - \omega t, x)$ *for every* $(x,t) \in [-\varepsilon^{-(2N+1)}, \varepsilon^{-(2N+1)}] \times (0, \infty)$ *with* $|x| + t < \varepsilon^{-2N+1}$.

Spatial dynamics formulation

Starting with the wave equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) + \rho(x) u(x,t) = \gamma u(x,t)^3, \quad \rho(x+2\pi) = \rho(x),$$

we introduce three spatial scales in

$$u(x,t) = v(\xi, z, x) \quad \text{with} \;\; \xi = x - c_g t, \;\; z = l_0 x - \omega t.$$

This yields

$$\left[(c^2-1)\partial_\xi^2 + 2(c\omega - l_0)\partial_\xi \partial_z - 2\partial_\xi \partial_x + (\omega^2 - l_0^2)\partial_z^2 - 2l_0 \partial_z \partial_x - \partial_x^2\right] v \\ + \rho(x) v = \gamma v^3,$$

with $v(\xi, z+2\pi, x) = v(\xi, z, x+2\pi) = v(\xi, z, x)$. We can use the Fourier series in z but not in x.

Spatial dynamics formulation

By using Fourier series in z and writing the first-order system in ξ, we obtain the spatial dynamical system:

$$(1-c^2)\partial_\xi \begin{pmatrix} \tilde{v}_m \\ \tilde{w}_m \end{pmatrix} = A_m(\omega, c) \begin{pmatrix} \tilde{v}_m \\ \tilde{w}_m \end{pmatrix} - \gamma \begin{pmatrix} 0 \\ (\tilde{v} * \tilde{v} * \tilde{v})_m \end{pmatrix}, \quad m \in \mathbb{Z},$$

where

$$A_m(\omega, c) = \begin{pmatrix} 0 & 1 \\ -(\partial_x + \mathrm{i}ml_0)^2 + \rho(x) - m^2\omega^2 & 2\mathrm{i}mc\omega - 2(\partial_x + \mathrm{i}ml_0) \end{pmatrix}.$$

For each $m \in \mathbb{Z}$, $A_m(\omega, c) : D \subset R \to R$ are linear operators with

$$D = H^2_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T}), \qquad R = H^1_{\rm per}(\mathbb{T}) \times L^2(\mathbb{T})$$

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We are looking for the solution map
$[0, \xi_0] \ni \xi \mapsto (\tilde{v}_m, \tilde{w}_m)_{m\in\mathbb{Z}} \in C^1([0, \xi_0], \mathcal{D})$ in function space

$$\mathcal{D} := [\ell^{2,2}(\mathbb{Z}, L^2(\mathbb{T})) \cap \ell^{2,1}(\mathbb{Z}, H^1_{\rm per}(\mathbb{T})) \cap \ell^{2,0}(\mathbb{Z}, H^2_{\rm per}(\mathbb{T}))] \times [\ell^{2,1}(\mathbb{Z}, L^2(\mathbb{T})) \cap \ell^{2,0}(\mathbb{Z}, H^1_{\rm per}(\mathbb{T}))].$$

Eigenvalues of the spatial system

Recall that the bifurcation case corresponds to $\omega_0 = \omega_{n_0}(l_0)$ and $c_g = \omega'_{n_0}(l_0)$. The eigenvalue problem $A_m(\omega_0, c_g)\vec{V} = \lambda\vec{V}$ is reformulated in the scalar form:

$$[-(\partial_x + \mathrm{i}ml_0 + \lambda)^2 + \rho(x)]V(x) = (m\omega_0 - \mathrm{i}c_g\lambda)^2 V(x),$$

which is solved with Bloch eigenfunctions in

$$\omega_n^2(ml_0 - \mathrm{i}\lambda) = (m\omega_0 - \mathrm{i}c_g\lambda)^2, \qquad n \in \mathbb{N}.$$

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$$\omega_n^2(ml_0 - \mathrm{i}\lambda) = (m\omega_0 - \mathrm{i}c_g\lambda)^2, \qquad n \in \mathbb{N}.$$

No information on roots of λ is available, but zero roots $\lambda = 0$ are controlled from the non-resonance conditions $\omega_n(l_0) \neq \omega_0$, $n \neq n_0$,

$$\omega_n^2(ml_0) \neq m^2\omega_0^2, \quad m \in \{3, 5, \dots 2N+1\}, \quad \forall n \in \mathbb{N}.$$

The zero root $\lambda = 0$ is double in the subspace $n = n_0$.

Eigenvalues of the spatial system

Recall that the bifurcation case corresponds to $\omega_0 = \omega_{n_0}(l_0)$ and $c_g = \omega'_{n_0}(l_0)$. The eigenvalue problem $A_m(\omega_0, c_g)\vec{V} = \lambda\vec{V}$ is reformulated in the scalar form:

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which is solved with Bloch eigenfunctions in

$$\omega_n^2(ml_0 - \mathrm{i}\lambda) = (m\omega_0 - \mathrm{i}c_g\lambda)^2, \qquad n \in \mathbb{N}.$$

One can show that the non-resonance conditions can be satisfied for $\rho(x) = 1$ (low-contrast potentials). In this case, the roots are defined by the quadratic equations

$$1 + (n + ml_0 - i\lambda)^2 = (m\omega_0 - ic_g\lambda)^2.$$

Moreover, one can find conditions when all roots are simple.

Algorithm for justification of a homoclinic orbit

Step 1: Decomposition near the bifurcation.

$$\begin{pmatrix} \tilde{v}_1(\xi,x) \\ \tilde{w}_1(\xi,x) \end{pmatrix} = \underbrace{\varepsilon q_0(\xi) F_0(x) + \varepsilon q_1(\xi) F_1(x)} + \varepsilon S_1(\xi,x),$$

and

$$\begin{pmatrix} \tilde{v}_m(\xi,x) \\ \tilde{w}_m(\xi,x) \end{pmatrix} = \varepsilon S_m(\xi,x), \quad m \neq 1,$$

where the small parameter is defined for $\omega = \omega_0 + \varepsilon^2$ and $c = c_g$.

Algorithm for justification of a homoclinic orbit

Step 2: Near-identity transformation to reduce the residual terms.

They are performed based on the bounds

$$\|(\Pi A_1(\omega_0, c_g)\Pi)^{-1}\|_{R\to D} + \sum_{m=3}^{2N+1} \|A_m(\omega_0, c_g)^{-1}\|_{R\to D} \leq C_0,$$

which is obtained from the resolvent equations

$$\begin{pmatrix} 0 & 1 \\ L_m & M_m \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

with

$$L_m = -(\partial_x + \mathrm{i}ml_0)^2 + \rho(x) - m^2\omega_0^2,$$
$$M_m = 2\mathrm{i}mc_g\omega_0 - 2(\partial_x + \mathrm{i}ml_0),$$

Algorithm for justification of a homoclinic orbit

After Steps 1 and 2, the system

$$\frac{d}{d\xi}\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} q_1 \\ 0 \end{pmatrix} + \varepsilon^2 F(q_0, q_1, \mathbf{S})$$

$$\frac{d}{d\xi} S_m = A_m(\omega_0, c_g) S_n + \varepsilon^2 F_m(q_0, q_1, \mathbf{S})$$

becomes

$$\frac{d}{d\xi}\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} q_1 \\ 0 \end{pmatrix} + \sum_{j=1}^{N} \varepsilon^{2j} F^{(j)}(q_0, q_1) + \varepsilon^{2N+2} F^{(N)}(q_0, q_1, \mathbf{S})$$

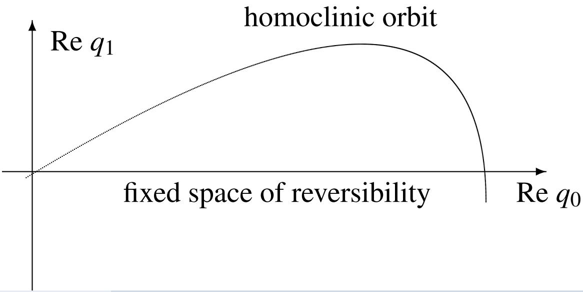
$$\frac{d}{d\xi} S_m = A_m(\omega_0, c_g) S_n + \varepsilon^{2N+2} F_m(q_0, q_1) + \varepsilon^2 \tilde{F}_m(q_0, q_1, \mathbf{S})$$

Algorithm for justification of a homoclinic orbit

Step 3: Construction of a reversible homoclinic orbit

$$\frac{d}{d\xi}\left(\begin{array}{c} q_0 \\ q_1 \end{array}\right) = \left(\begin{array}{c} q_1 \\ 0 \end{array}\right) + \sum_{j=1}^{N} \varepsilon^{2j} F^{(j)}(q_0, q_1)$$

satisfying $\text{Im}(q_0) = 0$ and $\text{Re}(q_1) = 0$.



Algorithm for justification of a homoclinic orbit

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satisfying $\mathrm{Im}(q_0) = 0$ and $\mathrm{Re}(q_1) = 0$.

We have the leading-order approximation with

$$\|q_0 - A(\varepsilon\cdot)\|_{L^\infty} \leq C\varepsilon, \quad \|q_1 - \varepsilon A'(\varepsilon\cdot)\|_{L^\infty} \leq C\varepsilon^2,$$

The persistence analysis is done by the implicit function theorem in $H^1(\mathbb{R})$ because of the symmetries of the truncated system with the 2-parameter family of solutions

$$(q_0(\xi+\xi_0)e^{i\theta_0}, q_1(\xi+\xi_0)e^{i\theta_0}), \quad \xi_0, \theta_0 \in \mathbb{R}.$$

Algorithm for justification of a homoclinic orbit

After Step 3, we can write $(q_0, q_1) = (Q_0, \varepsilon Q_1) + (\mathfrak{q}_0, \varepsilon \mathfrak{q}_1)$, where $(Q_0, \varepsilon Q_1)$ is the homoclinic orbit of the truncated system. The abstract system is

$$
\begin{aligned}
\partial_\xi \mathbf{c}_{0,r} &= \varepsilon \Lambda_0(\xi) \mathbf{c}_{0,r} + \varepsilon \mathbf{G}(\mathbf{c}_{0,r}, \mathbf{c}_r) + \epsilon^{2N+1} \mathbf{G}_R(\mathbf{c}_{0,hom} + \mathbf{c}_{0,r}, \mathbf{c}_r), \\
\partial_\xi \mathbf{c}_r &= \Lambda_r \mathbf{c}_r + \varepsilon^2 \mathbf{F}(\mathbf{c}_{0,hom} + \mathbf{c}_{0,r}, \mathbf{c}_r) + \varepsilon^{2N+2} \mathbf{F}_R(\mathbf{c}_{0,hom} + \mathbf{c}_{0,r}, \mathbf{c}_r),
\end{aligned}
$$

Λ_r contains nonzero eigenvalues for stable, center, and unstable manifolds of the linearized system. We assume

$$
\begin{aligned}
\|e^{\Lambda_s \xi}\|_{\mathcal{D} \to \mathcal{D}} \leq K, \qquad & \xi \geq 0, \\
\|e^{\Lambda_u \xi}\|_{\mathcal{D} \to \mathcal{D}} \leq K, \qquad & \xi \leq 0, \\
\|e^{\Lambda_c \xi}\|_{\mathcal{D} \to \mathcal{D}} \leq K, \qquad & \xi \in \mathbb{R}.
\end{aligned}
$$

Algorithm for justification of a homoclinic orbit

After Step 3, we can write $(q_0, q_1) = (Q_0, \varepsilon Q_1) + (\mathfrak{q}_0, \varepsilon \mathfrak{q}_1)$, where $(Q_0, \varepsilon Q_1)$ is the homoclinic orbit of the truncated system. The abstract system is

$$\begin{aligned} \partial_\xi \mathbf{c}_{0,r} &= \varepsilon \Lambda_0(\xi) \mathbf{c}_{0,r} + \varepsilon \mathbf{G}(\mathbf{c}_{0,r}, \mathbf{c}_r) + \epsilon^{2N+1} \mathbf{G}_R(\mathbf{c}_{0,hom} + \mathbf{c}_{0,r}, \mathbf{c}_r), \\ \partial_\xi \mathbf{c}_r &= \Lambda_r \mathbf{c}_r + \varepsilon^2 \mathbf{F}(\mathbf{c}_{0,hom} + \mathbf{c}_{0,r}, \mathbf{c}_r) + \varepsilon^{2N+2} \mathbf{F}_R(\mathbf{c}_{0,hom} + \mathbf{c}_{0,r}, \mathbf{c}_r), \end{aligned}$$

Step 4: Center-stable manifold. For every $\mathbf{a} \in \mathcal{D}_c$, $\mathbf{b} \in \mathcal{D}_s$ s.t. $\|\mathbf{a}\|_{\mathcal{D}_c} + \|\mathbf{b}\|_{\mathcal{D}_s} \leq C\varepsilon^{2N}$, there exists a family of local solutions with

$$\sup_{\xi \in [0, \varepsilon^{-(2N+1)}]} \left(\|\mathbf{c}_{0,r}(\xi)\|_{\mathbb{C}^4} + \|\mathbf{c}_c(\xi)\|_{\mathcal{D}_c} + \|\mathbf{c}_s(\xi)\|_{\mathcal{D}_s} + \|\mathbf{c}_u(\xi)\|_{\mathcal{D}_u} \right) \leq C\varepsilon^{2N},$$

satisfying $\mathbf{c}_c(0) = \mathbf{a}$ and $e^{-\xi_0 \Lambda_s} \mathbf{c}_s(\xi_0) = \mathbf{b}$ at $\xi_0 = \varepsilon^{-(2N+1)}$. These parameters are chosen to satisfy the reversibility constraints.

Section 4

Breathers localized in time

Example: the focusing NLS equation

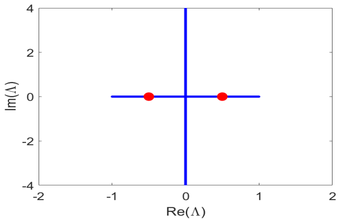
The focusing nonlinear Schrödinger (NLS) equation

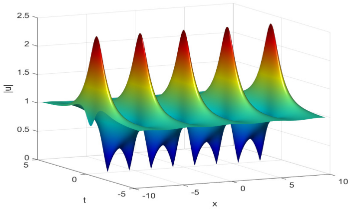
$$i\partial_t \psi + \partial_x^2 \psi + |\psi|^2 \psi = 0$$

admits the exact solution [Akhmediev, Eleonsky, & Kulagin (1985)]

$$\psi(x,t) = e^{it} \left[1 - \frac{2(1-\lambda^2)\cosh(k\lambda t) + ik\lambda \sinh(k\lambda t)}{\cosh(k\lambda t) - \lambda \cos(kx)} \right],$$

commonly known as Akhmediev breathers.





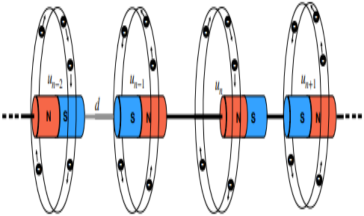
The engineering setup

The FPU model:

$$\underline{m}\ddot{u}_n + k(t)u_n = \beta(d + u_n - u_{n-1})^{-\alpha} - \beta(d + u_{n+1} - u_n)^{-\alpha},$$

where $\alpha, \beta, \underline{m}, d > 0$ and $k(t + 2\pi) = k(t)$.

FPU models a chain of repelling magnets surrounded by time modulated coils (Chong, Kim, Daraios et al.: arXiv:2310.06934)



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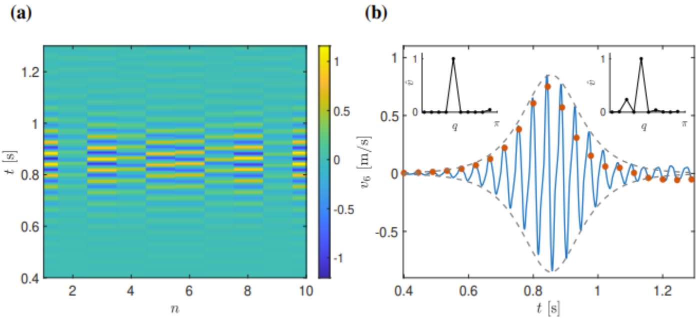
Time-localized breathers were observed in experiments:



Bifurcation theory

For N particles with Dirichlet conditions $u_0 = u_{N+1} = 0$, we use the discrete Fourier sine modes:

$$u_n(t) = \sum_{m=1}^{N} \hat{u}_m(t) \sin(q_m n), \quad q_m := \frac{\pi m}{N+1}, \quad 1 \leq m \leq N$$

and obtain the linear Schrodinger problem

$$\mathcal{L} \hat{u}_m = \lambda_m \hat{u}_m, \qquad \mathcal{L} = -\underline{m} \partial_t^2 - k(t),$$

where $\lambda_m = 4 \sin^2\left(\frac{q_m}{2}\right)$.

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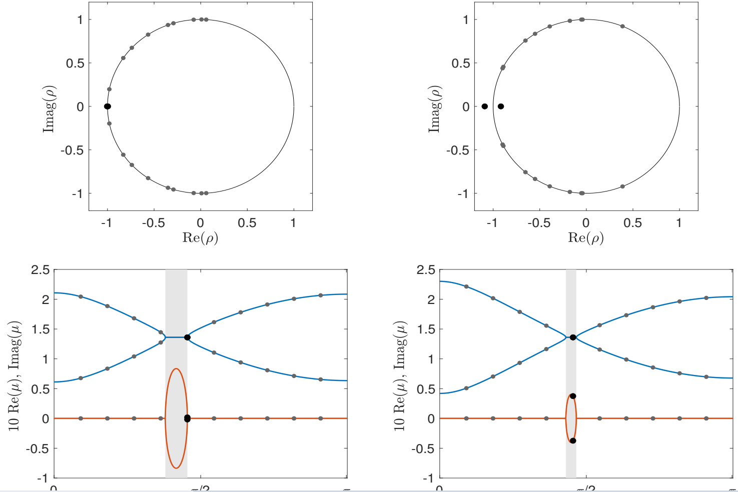
The spectrum of $\mathcal{L}$ is purely continuous in

$$\sigma(\mathcal{L}) = [\nu_0, \mu_1] \cup [\mu_2, \nu_1] \cup [\nu_2, \mu_3] \cup [\mu_4, \nu_3] \cup \cdots$$

We are looking for a bifurcation case of $k_0(t)$ when $\lambda_{m_0} = \mu_1$ or $\lambda_{m_0} = \mu_2$ for one $m_0 \in \{1, 2, \dots, N\}$.



Main theorem [C. Chong, D.P., G. Schneider (2024)]

Assume two conditions (spectral assumption and nonzero normal form). Then there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and every $M \in \mathbb{N}$ the FPU system possesses two generalized homoclinic solutions $U^{\pm}_{\rm hom}(t) : [-\varepsilon^{-M+1}, \varepsilon^{-M+1}] \to \mathbb{R}^N$ satisfying

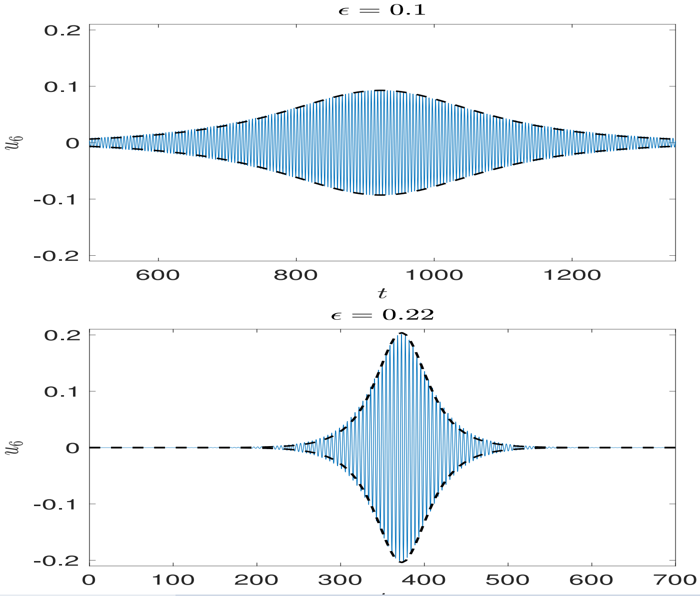
$$\sup_{t \in [-\varepsilon^{-M+1}, \varepsilon^{-M+1}]} \|U^{\pm}_{\rm hom}(t) - \mathcal{U}^{\pm}(t)\| + \|(U^{\pm}_{\rm hom})'(t) - (\mathcal{U}^{\pm})'(t)\| \leq C\varepsilon^{M-1}$$

where $\mathcal{U}^{\pm}(t) : \mathbb{R} \to \mathbb{R}^N$ satisfy $\lim\limits_{|t| \to \infty} \|\mathcal{U}^{\pm}(t)\| + \|(\mathcal{U}^{\pm})'(t)\| = 0$ and can be approximated as

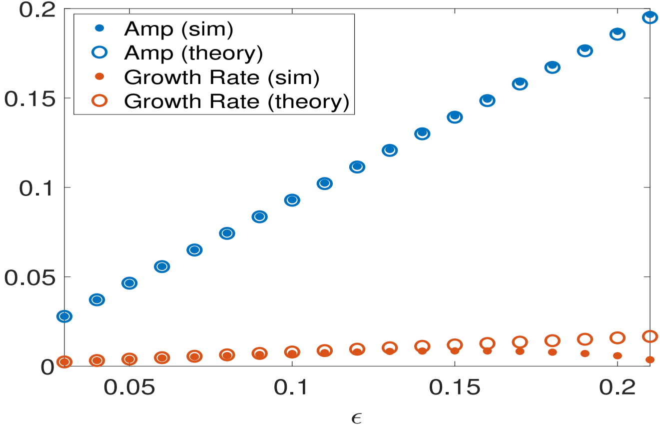
$$(\mathcal{U}^{\pm})_n(t) = \pm\varepsilon \left[A(\varepsilon t)\mathcal{F}(t) + \bar{A}(\varepsilon t)\overline{\mathcal{F}}(t)\right] \sin(q_{m_0} n) + \mathcal{O}(\varepsilon^2),$$

where $\mathcal{F}(t + T) = -\mathcal{F}(t)$ and $A(\tau) = \alpha \text{sech}(\beta\tau)$ are uniquely defined with some $\alpha, \beta > 0$.

Numerical illustration



Comparison between normal form and numerics



Algorithm for justification of the homoclinic solutions

Step 1: Bifurcation setup.

Let $k(t) = k_0(t) + \sigma\varepsilon^2$ and pick $k_0(t)$ so that $\lambda_{m_0} = \mu_1$ for one $m_0 \in \{1, 2, \dots, N\}$. This corresponds to the spectral band $\{\lambda_1(\ell)\}_{\ell \in [0, \frac{2\pi}{T})}$ with $\lambda_1(\ell_0) = \mu_1$ for $\ell_0 = \frac{\pi}{T}$. Assume no other Floquet multipliers to coincide with $+1$ or -1.

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Step 2: Formal derivation of the normal form. Expanding

$$u_n(t) = \varepsilon U_n^{(1)}(t) + \varepsilon^2 U_n^{(2)}(t) + \varepsilon^3 U_n^{(3)}(t) + \mathcal{O}(\varepsilon^4),$$

we select the leading order in the form

$$U_n^{(1)}(t) = A(\varepsilon t) g_1(t) \sin(q_{m_0} n),$$

where $g_1(t+T) = -g_1(t)$ is the bifurcating mode of $\mathcal{L}_0 g_1 = \mu_1 g_1$.

Algorithm for justification of the homoclinic solutions

At the order of $\mathcal{O}(\varepsilon^2)$, we get

$$\mathcal{L}_0 U_n^{(2)} + \Delta U_n^{(2)} = 2\underline{m} A'(\tau) g_1'(t) \sin(q_{m_0} n) + \chi_2 A(\tau)^2 g_1(t)^2 F_n^{(2)},$$

where $\tau = \varepsilon t$ and $F_n^{(2)} = -2\sin(q_{m_0})(1-\cos(q_{m_0}))\sin(2q_{m_0} n)$.

The solution for $U_n^{(2)}(t)$ can be written in the form

$$U_n^{(2)}(t) = A'(\tau) h_1(t) \sin(q_{m_0} n) + \chi_2 A(\tau)^2 h_2(t) \sin(2q_{m_0} n),$$

where

$$(\mathcal{L}_0 - \omega^2(q_{m_0}))h_1 = 2\underline{m} g_1'(t),$$

$$(\mathcal{L}_0 - \omega^2(2q_{m_0}))h_2 = -2\sin(q_{m_0})(1-\cos(q_{m_0}))g_1(t)^2.$$

The unique solution for $h_1(t+T) = -h_1(t)$ and $h_2(t+T) = h_2(t)$ exists under the spectral assumption.

Algorithm for justification of the homoclinic solutions

At the order of $\mathcal{O}(\varepsilon^3)$, we get

$$\mathcal{L}_0 U_n^{(3)} + \Delta U_n^{(3)} = \sigma U_n^{(1)} + 2m\partial_\tau \partial_t U_n^{(2)} + m\partial_\tau^2 U_n^{(1)}$$
$$+ 2\chi_2 \left[(U_{n+1}^{(1)} - U_n^{(1)})(U_{n+1}^{(2)} - U_n^{(2)}) - (U_n^{(1)} - U_{n-1}^{(1)})(U_n^{(2)} - U_{n-1}^{(2)}) \right]$$
$$- \chi_3 \left[(U_{n+1}^{(1)} - U_n^{(1)})^3 - (U_n^{(1)} - U_{n-1}^{(1)})^3 \right].$$

Projection to the mode $\sin(q_{m_0} n)$ yields the cubic normal form:

$$\frac{1}{2}\lambda_1''(\ell_0) A''(\tau) + \sigma A(\tau) + \chi A(\tau)^3 = 0,$$

where $\lambda_1''(\ell_0)$ is the band curvature at $\lambda_1(\ell_0) = \mu_1$, where $\lambda_1'(\ell_0) = 0$, and $\chi \neq 0$ under the normal form assumption.

Algorithm for justification of the homoclinic solutions

Step 3: Justification of the normal form. The normal form theorem near the double period bifurcation (Iooss–Adelmeyer, 1998) after diagonalization, near-identity transformations, and the use of reversibility.

Conclusion

- Generalized breathers have been considered either as the time-periodic and space-localized pulses or as the time-localized and space-periodic orbits.

- These solutions can be recovered in the spatial dynamical systems on a long but finite spatial scale.

- Numerical experiments do not often distinguish between true breathers and generalized modulating pulses.

MANY THANKS FOR YOUR ATTENTION!

BEST WISHES TO MICHAEL!!!