Localized Travelling Waves in Nonlinear Schrödinger Lattices

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Model

- Discrete nonlinear Schrödinger equation (DNLS) in 1-D

\[ i\dot{u}_n(t) + \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{\hbar^2} + f(u_{n+1}, u_n, u_{n-1}) = 0. \]

- General nonlinear term \( f \):
  - Cubic DNLS, \( f = |u_n|^2 u_n \).
  - Ablowitz-Ladik \( f = |u_n|^2 (u_{n+1} + u_{n-1}) \).
  - Salerno model

\[ f = 2\alpha |u_n|^2 u_n + (1 - \alpha) |u_n|^2 (u_{n+1} + u_{n-1}). \]

- Translationally invariant model

\[ f = \alpha_1 |u_n|^2 u_n + \alpha_2 |u_n|^2 (u_{n+1} + u_{n-1}) + \alpha_3 u_n^2 (\bar{u}_{n+1} + \bar{u}_{n-1}) + \ldots + \alpha_{10} (|u_{n+1}|^2 u_{n-1} + |u_{n-1}|^2 u_{n+1}). \]
More on translationally invariant model

- Stationary solutions $u_n(t) = \phi_n e^{i\omega t}$ satisfy the second-order difference map

$$-\omega \phi_n + \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} + f(\phi_{n+1}, \phi_n, \phi_{n-1}) = 0.$$  

- Two solutions: on-site and inter-site discrete solitons

- When $\alpha_1 = \alpha_4 + \alpha_6$, $\alpha_5 = \alpha_6$, $\alpha_7 = \alpha_4 - \alpha_6$ and $\alpha_{10} = \alpha_8 - \alpha_9$, the difference map admits a continuous family of localized solutions $\phi_n = \phi(n - s)$, where $s \in \mathbb{R}$ (D.P., Nonlinearity 19, 2695 (2006)).
Traveling waves in lattices

- **Discrete nonlinear Schrödinger equation**

\[
i\dot{u}_n(t) + \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{h^2} + f(u_{n+1}, u_n, u_{n-1}) = 0.
\]

- Moving into the travelling frame \(z = hn - 2ct\) gives a differential advance-delay equation. If \(u_n(t) = \phi(z)e^{i\omega t}\),

\[
2ic\phi'(z) = \frac{\phi(z + h) - 2\phi(z) + \phi(z - h)}{h^2} - \omega\phi(z)
\]

\[
+ f(\phi(z + h), \phi(z)\phi(z - h)).
\]

- Traveling waves satisfy the constraints:

\[
u_1(t) = u_0(t - \tau)e^{i\theta}, \quad u_2(t) = u_0(t - 2\tau)e^{2i\theta}, \quad \text{etc.}
\]
Radiationless Solitons

- Localised solutions to a differential difference equation.
- Waves travel across a lattice without shedding any radiation.
- Homoclinic orbit to the zero state in a travelling frame.
Difficulties

- In general, traveling wave solutions are weakly non-local.
- Eigenvalues on the imaginary axis in the linear spectrum give rise to radiation modes.
- Number of eigenvalues is finite for $c \neq 0$ but increases as $c \to 0$.
- In general there is at least one resonance.
- Amplitude of radiation modes are generally exponentially small in terms of a bifurcation parameter.
Reformulation of existence problem

- Introduce parameters $\kappa \in \mathbb{R}_+, \beta \in [0, \pi]$
  \[
  \omega = \frac{2}{h} \beta c + \frac{2}{h^2} (\cos(\beta) \cosh(\kappa) - 1),
  \]
  \[
  c = \frac{1}{h\kappa} \sin(\beta) \sinh(\kappa),
  \]

- Scale out $h$ using $\phi(z) = \frac{1}{h} \Phi(Z) e^{i\beta Z}, Z = \frac{z}{h}$

- New differential advance-delay equation
  \[
  i \sin(\beta) \left( 2 \frac{\sinh(\kappa)}{\kappa} \frac{d\Phi(Z)}{dZ} - \Phi(Z + 1) + \Phi(Z - 1) \right)
  \]
  \[
  + \cos(\beta) \left( 2 \cosh(\kappa) \Phi(Z) - \Phi(Z + 1) - \Phi(Z - 1) \right)
  \]
  \[
  - f(\Phi(Z + 1)e^{i\beta}, \Phi(Z), \Phi(Z - 1)e^{-i\beta}) = 0,
  \]
  where $\kappa > 0$ and $\beta \in [0, \pi]$. 

Linear Spectrum

- Dispersion relation for the linear equation is obtained using
  \[ \Phi(Z) = e^{pZ} \]

\[ D(p; \kappa, \beta) = 2 \cos(\beta)(\cosh(p) - \cosh(\kappa)) + 2i \sin(\beta) \left( \sinh(p) - \frac{\sinh(\kappa)}{\kappa} p \right) = 0. \]

- there are finitely many imaginary roots \( p = ik_n, \ n = 1, \ldots, m \) for any \( \kappa > 0 \) and \( \beta \in (0, \pi) \)
- if \( \kappa = 0 \), there exists a double root \( k = 0 \) of \( D(ik; 0, \beta) \)
- if \( \kappa = 0 \) and \( \beta = \pi/2 \), the zero root \( k = 0 \) is a triple root of \( D(ik; 0, \beta) \)
- if \( \kappa = 0 \) and \( \beta \in (\beta_0, \pi/2) \) with \( \beta_0 \approx \frac{\pi}{13} \), there exists only one imaginary root besides the double zero root.
• Dispersion relation for the linear equation is obtained using
\[ \phi(Z) = e^{\rho Z} \]
Methods

- Normal forms and Melnikov integrals
  - Analysis of the normal form near $\kappa = 0$ and $\beta = \pi/2$ (D.P., V. Rothos, Physica D 202, 16 (2005)).
  - Analysis of persistence of homoclinic orbits near the line $\kappa > 0$ and $\beta = \pi/2$ (D.P., T. Melvin, A. Champneys, Physica D 236, 22 (2007)).
Methods

- **Stokes constant computation**
- Analysis of Stokes phenomena in a beyond all orders expansion for $\kappa = 0$ and $\beta \neq \pi/2$ (O. Oxtoby, I. Barashenkov, nlin/0610059 (2006)).
Methods

- Pseudo-spectral decomposition
  - Numerical solutions of the differential advance-delay equation for $\kappa > 0$ and any $\beta$. 
Reduction of the differential advance-delay equation

- Write the main equation as

\[
i \sin(\beta) \left( \Phi_+ - \Phi_- - 2 \frac{\sinh(\kappa)}{\kappa} \Phi'(Z) \right) + \cos(\beta) (\Phi_+ + \Phi_- - 2 \cosh(\kappa) \Phi) + f_r + if_i = 0,\]

where \( \Phi_\pm = \Phi(Z \pm 1) \) and \( f(\Phi_+ e^{i\beta}, \Phi, \Phi_- e^{-i\beta}) = f_r + if_i \).

- If \( \beta = \frac{\pi}{2} \) and

\[
\alpha_1 = 0, \quad \alpha_4 = \alpha_6, \quad \alpha_7 = 2 \alpha_5,
\]

the equation reduces to a scalar real-valued equation

\[
2 \frac{\sinh(\kappa)}{\kappa} \frac{d\Phi}{dZ} = \left[ 1 + (\alpha_2 - \alpha_3) \Phi^2 + \alpha_8 (\Phi_+^2 + \Phi_+ \Phi_- + \Phi_-^2) - (\alpha_9 + \alpha_{10}) \Phi_+ \Phi_- \right] (\Phi_+ - \Phi_-).
\]
Assumption on existence of solutions

• **Assumption:** There exists a single-humped solution $\Phi_0(Z)$ for any $\kappa > 0$ and some parameters $(\alpha_2^{(0)}, \alpha_3^{(0)}, \ldots)$ s.t.

$$\Phi_0 \in H^1(\mathbb{R}) : \Phi_0(-Z) = \Phi_0(Z), \quad \lim_{|Z| \to \infty} e^{\kappa|Z|} \Phi_0(Z) = c_0.$$ 

• Any even solution is extended into a continuous family $\Phi_0(Z - s), \forall s \in \mathbb{R}$.

• When $\alpha_8 = \alpha_9 = \alpha_{10} = 0$ and $\alpha_2 > \alpha_3$, the assumption is satisfied with the explicit solution

$$\Phi_0(Z) = \frac{\sinh \kappa}{\sqrt{\alpha_2 - \alpha_3}} \sech(\kappa Z), \quad \kappa > 0.$$
Theorem: Under some assumptions on the linearized operator, the single-humped localized solution persists with respect to parameter continuations, such that \( \| \Phi - \Phi_0 \|_{H^1} \leq C \epsilon \), where \( C > 0 \) and \( \epsilon = \max_j |\alpha_j - \alpha_j^{(0)}| \).

To the proof:

- Let \( \Phi = \Phi_0 + U \), \( \alpha_j = \alpha_j^{(0)} + \epsilon a_j \), and write the scalar equation as

\[
L_+ U = N(U) + \epsilon F(\Phi_0 + U),
\]

where \( L_+ \) is a differential advance-delay operator and

\[
\| N(U) \|_{H^1} \leq C_1 \| U \|_{H^1}^2, \quad \| F(\Phi_0 + U) \|_{H^1} \leq C_2 \| \Phi_0 + U \|_{H^1}^3.
\]
To the proof:

- Notice that

\[ L_+ : H^1_{\text{ev}} \mapsto L^2_{\text{odd}}, \quad N, F : H^1_{\text{ev}} \mapsto H^1_{\text{odd}} \]

and

\[ L_+ \Phi_0'(Z) = 0, \quad L_+ \frac{\partial \Phi_0}{\partial \kappa} = \frac{2(\kappa \cosh \kappa - \sinh \kappa)}{\kappa^2} \Phi_0'(Z). \]

- Assume that \( L_+ \) has no eigenvalues near \( \text{Re}(\lambda) = 0 \) except for \( \lambda = 0 \) and that the zero eigenvalue is double. Then, invert \( L_+ \) on \( L^2_{\text{odd}} \) and use the Implicit Function Theorem.

- Although the continuous spectrum of \( L_+ \) extends on the imaginary axis \( \text{Re}(\lambda) = 0 \), the entire spectrum is shifted off the imaginary axis in the exponentially weighted \( H^1 \) space.
Persistence of solutions II

Theorem: Under additional assumptions on the linearized operator, the single-humped localized solution persists along the curve on \((\kappa, \beta)\)-plane with respect to parameter continuations, such that \(\|\Phi - \Phi_0\|_{H^1} \leq C(\epsilon + \mu)\) if and only if \(\Delta(\epsilon, \mu) = 0\), where \(\mu = \cot \beta\), \(\alpha_j = \alpha_j^{(0)} + \epsilon a_j\) and \(\Delta(\epsilon, \mu)\) is a Melnikov integral

\[
\Delta(\epsilon, \mu) = \int_\mathbb{R} W_0(Z; 0)[N_-(U, V) + F_-(\Phi_0 + U, V; \epsilon, \mu)]dZ,
\]

where

- \(W_0\) is an eigenfunction of the adjoint operator for the zero eigenvalue,
- \(N_-\) is the unperturbed vector field with quadratic and cubic terms in \(\Phi(Z) - \Phi_0(Z) = U(Z) + iV(Z)\), and
- \(F_-\) contain linear and nonlinear terms in \(\Phi_0 + U\) and \(V\) related to the perturbations in \(\mu = \cot \beta\) and \(\epsilon\).

It is clear that \(\Delta(0, 0) = 0\).
Example I: Salerno model

The model is

\[ i\dot{u}_n(t) + \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{\hbar^2} + 2\alpha|u_n|^2u_n + (1 - \alpha)|u_n|^2(u_{n+1} + u_{n-1}) = 0. \]

- If \( \alpha = 0 \), the family of solutions with \( \beta = \frac{\pi}{2} \) is a part of a two-parameter family. \( \implies \Delta(0, \mu) = 0 \) for any \( \mu \in \mathbb{R} \).
- If \( \Delta(\epsilon, 0) \neq 0 \) for \( \epsilon \neq 0 \), the family can not be continued in \( \epsilon \).
- Explicit computation shows that

\[ \partial_\epsilon \Delta(0, 0) = \int_{\mathbb{R}} W_0(Z; 0)\Phi_0^3(Z)dZ \approx -\frac{\kappa^2}{2} \int_{\mathbb{R}} \frac{d\zeta}{\cosh^3 \zeta} < 0, \]

for small \( \kappa > 0 \).
- Therefore, the family of exact solutions of the AL lattice does not persist in the Salerno model near \( \beta = \frac{\pi}{2} \).
Example II: Translationaly invariant model

The model is

\begin{align*}
    i \dot{u}_n(t) &+ \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{h^2} \\
    &+ \alpha_1 |u_n|^2 u_n + \alpha_2 |u_n|^2 (u_{n+1} + u_{n-1}) + \alpha_3 u_n^2 (\bar{u}_{n+1} + \bar{u}_{n-1}) \\
    &+ \ldots + \alpha_{10} (|u_{n+1}|^2 u_{n-1} + |u_{n-1}|^2 u_{n+1}) = 0.
\end{align*}

The exact solution exists for \( \alpha_1 = \alpha_4 = \ldots = 0 \) and \( \alpha_2 > \alpha_3 \).

- If \( \partial_\mu \Delta(0, 0) \neq 0 \) for any \( \kappa > 0 \), there exists a unique continuation of the solution \( \Phi_0 \) near the line \( \beta = \frac{\pi}{2} \).
- Explicit computation shows that

\[
    \partial_\mu \Delta(0, 0) = 2\alpha_3 \int_{\mathbb{R}} W_0(Z; 0) \Phi^2(\Phi_+ + \Phi_-) dZ \\
    \approx 4\kappa^2 \alpha_3 \int_{\mathbb{R}} \left( 1 - 2 \text{sech}^2 \zeta \right) \text{sech}^3 \zeta d\zeta \neq 0,
\]

for small \( \kappa > 0 \).
Example II: Translationally invariant model

In addition,

\[ \partial_\epsilon \Delta(0,0) = \int_{\mathbb{R}} W_0(Z;0) \left[ \alpha_1 \Phi^3 + (\alpha_4 - \alpha_6) \Phi(\Phi^2_+ + \Phi^2_-) ight. \\
+ (\alpha_7 - 2\alpha_5) \Phi \Phi_+ \Phi_- \right] dZ, \]

which is zero for \( \alpha_1 = 0, \alpha_4 = \alpha_6, \) and \( \alpha_7 = 2\alpha_5. \)

The localized solution persists on the line \( \beta = \frac{\pi}{2} \) if

\[ \alpha_1 = 0, \quad \alpha_4 = \alpha_6, \quad \alpha_7 = 2\alpha_5. \]
Use a pseudo-spectral method to transform differential advance-delay equation $\rightarrow$ system of algebraic equations

$$\Phi(Z_i) = \sum_{j=1}^{N} a_j \cos \left( \frac{2\pi j}{L} Z_i \right) + i b_j \sin \left( \frac{2\pi j}{L} Z_i \right).$$

Solutions are defined on a large finite domain $L$ at the collocation points $Z_i = \frac{Li}{2(N+1)}$.

Solutions have generally a non-zero radiation tail near the end points $Z = \pm L/2$. To measure the tail, we use the signed amplitude

$$\Delta = \text{Im}(\Phi(L/2)).$$
Example I : Salerno model

The model is

\[ i\dot{u}_n(t) + \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{h^2} + 2\alpha |u_n|^2 u_n + (1 - \alpha) |u_n|^2 (u_{n+1} + u_{n-1}) = 0. \]

Localised solutions do not exist for \( \alpha = 0.9, 1.1, \beta = 0.35\pi, 0.65\pi \) (left) but do exist for \( \alpha = 0.7, \beta = 0.875\pi \) (right).
Example I: Salerno model

Profiles of solutions for real part of $\Phi(Z)$ versus tail amplitude $\Delta$ (left). Solution branches for a fixed $\kappa > 0$: one-humped for $\beta > \frac{\pi}{2}$ and two-humped for $\beta < \frac{\pi}{2}$ (right).
Example II : Translationally invariant model

The model is

\[ i \dot{u}_n(t) + \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{h^2} + \alpha_1 |u_n|^2 u_n + \alpha_2 |u_n|^2 (u_{n+1} + u_{n-1}) + \alpha_3 u_n^2 (\bar{u}_{n+1} + \bar{u}_{n-1}) \]

\[ \ldots + \alpha_{10} (|u_{n+1}|^2 u_{n-1} + |u_{n-1}|^2 u_{n+1}) = 0. \]

If \( \alpha_1 = 0, \alpha_4 = \alpha_6, \alpha_7 = 2\alpha_5 \), the solution persists for \( \beta = \frac{\pi}{2} \).
Example II: Translationaly invariant model

The solution persists generally as a one-parameter curve on the parameter plane.
Example II: Translationally invariant model

Branches of single-humped solutions connect to branches of double-humped solutions.
Conclusion

- Traveling localized waves are still generic in many discrete NLS equations in spite of the presence of resonances.

- One-parameter curves in non-integrable lattices are more structurally stable with respect to perturbations than two-parameter curves in near-integrable lattices.

- Traveling localized waves in the translationally invariant model are stable with respect to time-dependent perturbations.

- Salerno model also has traveling localized wave solutions (away from the integrable Ablowitz–Ladik limit).