Exponentially small splitting for heteroclinic and homoclinic orbits in lattice equations

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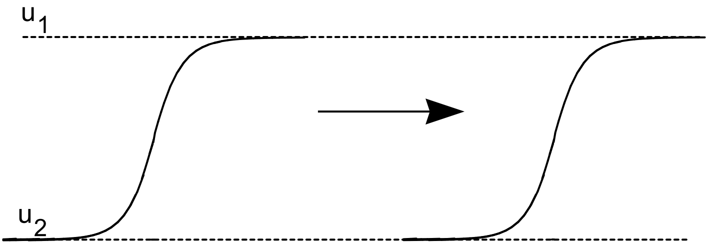
Nonlinear Klein–Gordon equation

1D case:

$u_{tt} - u_{xx} + V'(u) = 0$
where $V(u)$ is nonlinear potential (depends on a physical context)

Kink (domain wall) solutions:

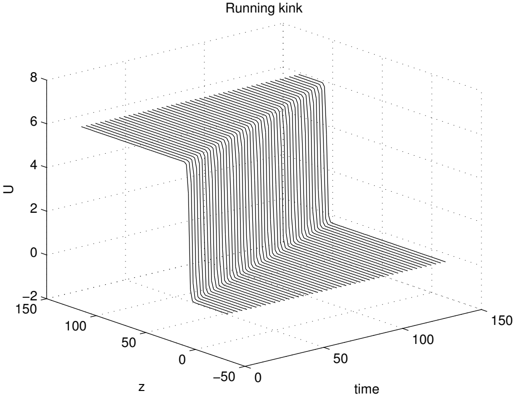
$\lim_{x\to-\infty} u(x,t) = u_2, \quad \lim_{x\to\infty} u(x,t) = u_1$



Nonlinear Klein–Gordon equation

Travelling waves: $u(x, t) = u(x - ct) \equiv u(z)$.

ODE: $(1 - c^2)u_{zz} - V'(u) = 0$

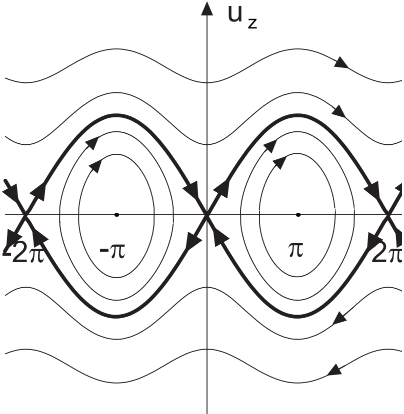


Nonlinear Klein–Gordon equation

Example 1: the sine-Gordon equation

$u_{tt} - u_{xx} + \sin u = 0.$

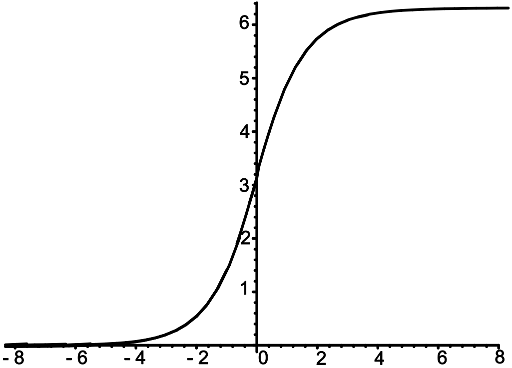
Travelling waves: $(1 - c^2)u_{zz} = \sin u.$



Nonlinear Klein–Gordon equation

- Only 2π-kink (antikink) solutions exist
- Solutions exist for arbitrary velocity c as long as $c^2 < 1$

$$u(z) = 4 \arctan \exp\left\{ \pm \frac{z - z_0}{\sqrt{1 - c^2}} \right\}, \quad z = x - ct.$$



Nonlinear Klein–Gordon equation

Example 2: the double sine-Gordon equation

$u_{tt} - u_{xx} + \sin u + 2a \sin 2u = 0.$

- Exact 2π-kink solution exist for $1 + 4a > 0$:

$$u(z) = \pi + 2 \arctan \left(\frac{1}{\sqrt{1+4a}} \sinh \left[\frac{\sqrt{1+4a}}{\sqrt{1-c^2}} (z - z_0) \right] \right), \quad z = x - ct$$

- Solution exist for arbitrary velocity c as long as $c^2 < 1$

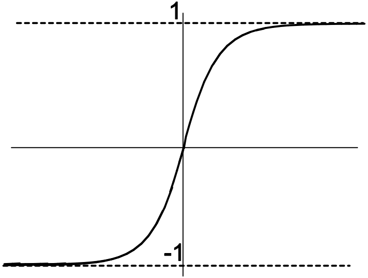
Nonlinear Klein–Gordon equation

Example 3: the ϕ^4 equation

$u_{tt} - u_{xx} - u + u^3 = 0.$

- Exact kink solution, exists for any $c^2 < 1$,

$$u(z) = \tanh\left(\frac{z - z_0}{\sqrt{2}\sqrt{1-c^2}}\right), \quad z = x - ct$$



Nonlinear Klein–Gordon equation

Example 4: the $\phi^4 - \phi^6$ equation

$u_{tt} - u_{xx} - u(1-u^2)(1+\gamma u^2) = 0.$

- Exact kink solution, exists for any $c^2 < 1$ and $\gamma > -1$:,

$$u(z) = \frac{\sqrt{18+6\gamma}\tanh\left(\frac{1}{2}\sqrt{2(1+\gamma)}\,(z-z_0)\right)}{\sqrt{18(1+\gamma) - 12\gamma\tanh^2\left(\frac{1}{2}\sqrt{2(1+\gamma)}\,(z-z_0)\right)}}, \quad z = \frac{x-ct}{\sqrt{1-c^2}}$$

Generic form:

$u_{tt} + \mathcal{L}u + V'(u) = 0$

- $\mathcal{L}$ is Fourier multiplier operator: $\widehat{\mathcal{L}u}(k) = P(k)\hat{u}(k)$
- $P(k)$ is the symbol of the operator $\mathcal{L}$
- If $P(k) = k^2$, we are back to the nonlinear Klein–Gordon equation.

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Applications of nonlocal Klein–Gordon equations:

- lattice models (solid state physics)
- complex dispersion (nonlinear optics)
- Josephson junctions (superconductivity).

Symbols:

- $P(k) = \frac{4}{\lambda^2}\sin^2\left(\frac{\lambda k}{2}\right)$ (Frenkel-Kontorova model, solid state physics)
- $P(k) = \frac{k^2}{1+\lambda^2 k^2}$ (Kac-Baker model, magnatic spin systems)
- $P(k) = \frac{k^2}{\sqrt{1+\lambda^2 k^2}}$ (Silin-Gurevich model, Josephson junctions)

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In all these cases: $P(k) \equiv P_\lambda(k)$ depends on λ and

$$P_\lambda(k) \to k^2 \quad \text{as} \quad \lambda \to 0.$$

As $\lambda \to 0$

$$u_{tt} + \mathcal{L}_\lambda u + V'(u) = 0 \quad \Rightarrow \quad u_{tt} - u_{xx} + V'(u) = 0$$

Main question:

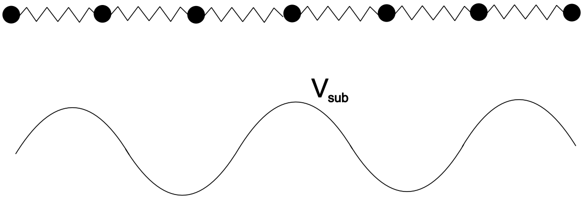
What happens with kink solutions when switching from local case $\lambda = 0$ to nonlocal case $\lambda \neq 0$?

The Frenkel-Kontorova model

Example 5: the Frenkel-Kontorova model (1938)

$u_{tt}(x,t) - \frac{1}{\lambda^2}(u(x+\lambda,t) - 2u(x,t) + u(x-\lambda,t)) + \sin u(x,t) = 0.$

describes a chain of particles with nearest-neighbours interactions.

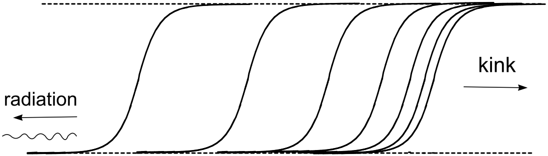


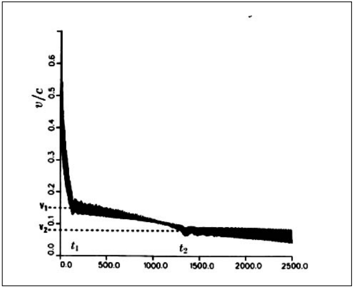
λ - a parameter of interaction between neighbours.

The symbol: $P(k) = \frac{4}{\lambda^2} \sin^2\left(\frac{\lambda k}{2}\right)$

The well-known (classical) results:

- There exist static 2π-kinks (on-site and inter-site).
- No travelling 2π-kinks.
- There exist infinitely many travelling 4π-kinks.
- A kink-like initial condition launched at some nonzero velocity emits radiation, slows down, and eventually stops.





(from M.Peyrard, M.D.Kruskal, Physica D, 14, p.88 (1984), initial velocity =0.8.)

Why do traveling kink solutions stop?

Consider linearized Frenkel-Kontorova model at zero equilibrium:

$$u_{tt}(x,t) - \tfrac{1}{\lambda^2}(u(x+\lambda,t) - 2u(x,t) + u(x-\lambda,t)) + u(x,t) = 0.$$

Dispersion relation for traveling waves after Fourier transform:

$$1 + \frac{4}{\lambda^2}\sin^2\left(\frac{\lambda k}{2}\right) = c^2 k^2, \quad k \in \mathbb{R},$$

For every $c \neq 0$, there exists at least one pair of solutions at $k = \pm k_0$.

SG equation with Kac-Baker interactions

Example 6: the sine-Gordon model with Kac-Baker interactions

$$u_{tt} - \frac{1}{2\lambda}\int_{-\infty}^{\infty} \exp\left(-\frac{|x-x'|}{\lambda}\right) u_{x'x'}(x',t)\; dx' + \sin u = 0.$$

SG equation with Kac-Baker interactions

Example 6: the sine-Gordon model with Kac-Baker interactions

$$u_{tt} - \frac{1}{2\lambda}\int_{-\infty}^{\infty} \exp\left(-\frac{|x-x'|}{\lambda}\right) u_{x'x'}(x',t)\ dx' + \sin u = 0.$$

This model is local since $q(x,t) = \dfrac{1}{2\lambda}\ \displaystyle\int_{-\infty}^{+\infty} \exp\left\{-\frac{|x-x'|}{\lambda}\right\} u(x',t)dx'$

satisfies $-\lambda^2 q_{xx} + q = u$.

The symbol: $P(k) = \dfrac{k^2}{1+\lambda^2 k^2}$

SG equation with Kac-Baker interactions

Travelling waves: $u(z) = u(x - ct)$

$$\boxed{\begin{aligned} & c^2 u_{zz} + \sin u = q_{zz} \\ & -\lambda^2 q_{zz} + q = u \end{aligned}}$$

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Phase space: $\{u \pmod{2\pi}, u', q, q'\}$

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Phase space: $\{u \pmod{2\pi}, u', q, q'\}$

Equilibrium points:
$O_0(u = u' = q = q' = 0)$, $O_\pi(u = q = \pi, u' = q' = 0)$

SG equation with Kac-Baker interactions

Travelling waves: $u(z) = u(x - ct)$

$$
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Equilibrium points:
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O_0 is the *saddle–center point*:

$$
1 + \frac{k^2}{1 + \lambda^2 k^2} = c^2 k^2
$$

For every $c \neq 0$, there exists exactly one pair of solutions at $k = \pm k_0$.

SG equation with Kac-Baker interactions

Results:

- There exist static 2π-kinks for $0 < \lambda < 1$.
- No travelling 2π-kinks.
- There exist infinitely many traveling 4π-kinks for a set of velocities.

SG equation with Kac-Baker interactions

Results:

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- No travelling 2π-kinks.
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Summary: switching from $\lambda = 0$ to $\lambda \neq 0$ results in disappearance of traveling 2π-kink solutions in lattice and nonlocal models.

Is this a general conclusion?

Main Claim

Consider the bifurcation problem in the general form

$$L_\lambda u + F(u) = 0.$$

- L_λ - a Fourier multiplier operator with an even symbol $P_\lambda(k)$ such that

$$P_\lambda(k) \to k^2 \text{ as } \lambda \to 0.$$

- $F(u)$ - an odd function such that $F(u_+) = F(u_-) = 0$ with $u_+ = -u_-$ and

$$F'(u_+) = F'(u_-) > 0$$

- Dispersion equation $P_\lambda(k) + F'(u_\pm) = 0$ has one pair of roots $k = \pm k_0(\lambda)$, such that $k_0(\lambda) \to \infty$ as $\lambda \to 0$.

Main Claim

Let us consider the limiting equation $u''(z) = F(u(z))$ and assume:

- It has an odd kink solution $u_0(z)$ for $z \in \mathbb{R}$ such that $u_0(z) \to u_\pm$ as $z \to \pm\infty$.

- When $u_0(z)$ is continued for $z \in \mathbb{C}$, the closest to real axis singularities are located in quartets, e.g. in the upper half-plane at $z_\pm = \pm\alpha + i\beta$, $\alpha, \beta > 0$.

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There exists an infinite set of values $\{\lambda_n\}_{n\in\mathbb{N}}$, such that for each λ_n, the nonlinear equation $L_{\lambda_n} u + F(u) = 0$ admits a kink solution. Moreover, the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfies the asymptotic law:

$$k_0(\lambda_n) \sim (n\pi + \varphi_0)/\alpha, \quad n \to \infty,$$

where φ_0 is uniquely defined constant.

Perturbation $v(z) = u(z) - u_0(z)$ satisfies the expanded equation

$$(L_\lambda + F'(u_0))\, v = H_\lambda + N(v),$$

where H_λ is the residual (explicitly computed from u_0) and $N(v)$ is $\mathcal{O}(v^2)$.

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- The homogeneous equation $(L_\lambda + F'(u_0))\, v = 0$ has a pair of solutions that behave like $e^{\pm i k_0(\lambda) z}$.

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- To satisfy the solvability condition at the leading order, we set

$$J_\pm(\lambda) := \int_{-\infty}^{\infty} e^{\pm i k_0(\lambda) z} H_\lambda(z) dz = 0$$

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- By Darboux principle and asymptotic analysis (Murray, 1984), if $H_\lambda(z) \sim C_0 \lambda^q e^{i\pi\kappa/2}(z - z_\pm)^\kappa$, then

$$J_\pm(\lambda) \sim \frac{4\pi\lambda^q |C_0| e^{-\beta k_0(\lambda)}}{\Gamma(-\kappa)|k_0(\lambda)|^{\kappa+1}} \cos(\alpha k_0(\lambda) + \pi/2 - \arg(C_0)).$$

Example 7: nonlocal double sine-Gordon model

$$u_{tt} - \frac{1}{2\lambda} \int_{-\infty}^{\infty} \exp\left(-\frac{|x - x'|}{\lambda}\right) u_{x'x'}(x')\, dx' = \sin(u) + 2a\sin(2u).$$

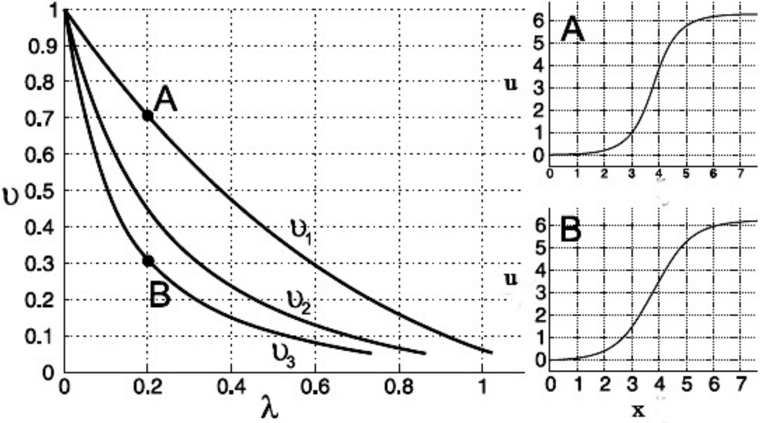
- As $\lambda \to 0$, the 2π-kinks are given by:

$$u(z) = \pi + 2\arctan\left(\frac{1}{\sqrt{1+4a}} \sinh\left[\frac{\sqrt{1+4a}}{\sqrt{1-c^2}}\,(z - z_0)\right]\right)$$

- Symmetric pairs of singularities exist for $a > 0$ at $z_{\pm} = \pm\alpha + i\beta$:

$$\alpha = \frac{\sqrt{1-c^2}}{2\sqrt{1+4a}} \cosh^{-1}(1+8a), \quad \beta = \frac{\pi\sqrt{1-c^2}}{2\sqrt{1+4a}}.$$

- For fixed $a > 0$, there exist a discrete set of curve in the (c, λ) plane, along which the 2π-kinks exist.



Three curves on the (c, λ) plane for $a = 1/8$.

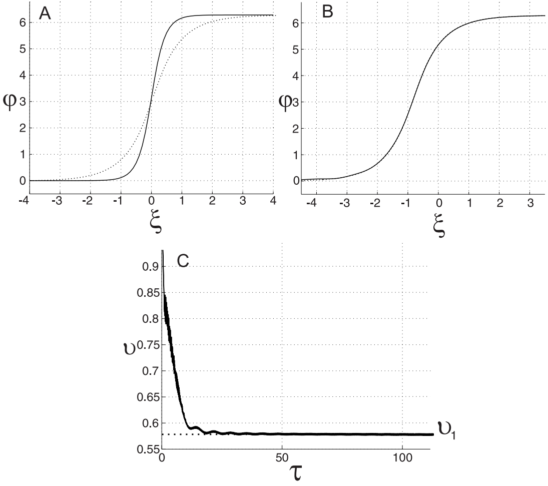
The asymptotic law as $n \to \infty$:

$$2\alpha k_0(\lambda_n) \sim \pi(1+2n), \quad \Rightarrow \quad \pi(1+2n)\lambda_n = \delta(a,c).$$

$1+2n$	1	3	5	7	9	11
$\delta/(\pi\lambda_n)$	3.7168	4.9763	6.3699	7.8595	9.4541	11.1396

Table: The values of $\delta/(\pi\lambda_n)$ for $a = 1/8$ and $c = 0.1$.

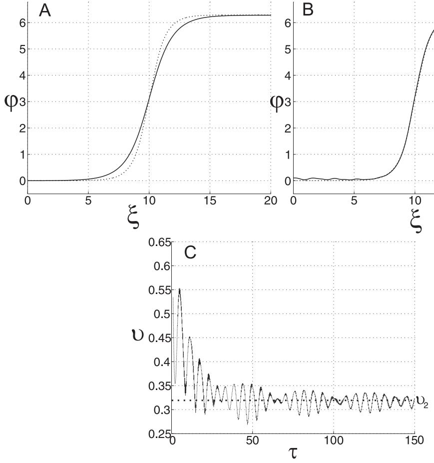
Numerical experiment 1 : initial speed is above 0.58



Evolution of kink-like excitation (high energy).

Nonlocal double SG model

Numerical experiment 2 : initial speed is below 0.58



Evolution of kink-like excitation (low energy).

Example 8: discrete ϕ^4 model

$u_{tt} - \lambda^{-2}(u(x+\lambda) - 2u(x) + u(x-\lambda)) + u(x)\left(1 - u(x)^2\right) = 0.$

- As $\lambda \to 0$, the kinks are given by:

$$u_0(z) = \tanh(\eta z), \qquad \eta = \frac{1}{2\sqrt{1-c^2}}.$$

- Singularity exists at $z = i\pi\sqrt{1-c^2}$.
- No kinks exist for any $c \neq 0$.

Example 9: discrete ϕ^4-ϕ^6 model

$u_{tt} - \lambda^{-2}(u(x+\lambda) - 2u(x) + u(x-\lambda)) + u(1-u^2)(1+\gamma u^2) = 0.$

- As $\lambda \to 0$, the kinks are given by:

$$u_0(z) = \frac{\sqrt{3+\gamma}\ \tanh(\eta z)}{\sqrt{3(1+\gamma) - 2\gamma\ \tanh^2(\eta z)}}, \qquad \eta = \frac{\sqrt{1+\gamma}}{\sqrt{2(1-c^2)}}.$$

- Symmetric pairs of singularities exist for $\gamma > 0$ at $z_\pm = \pm\alpha + i\beta$:

$$\alpha = \frac{\sqrt{1-c^2}}{2\sqrt{1+\gamma}} \cosh^{-1}\left(\frac{3+5\gamma}{3+\gamma}\right), \qquad \beta = \frac{\pi\sqrt{1-c^2}}{\sqrt{2(1+a)}}.$$

- For fixed $\gamma > 0$, there exist a discrete set of curve in the (c, λ) plane, along which the kinks exist.

The asymptotic law as $n \to \infty$:

$$4\alpha k_0(\lambda_n) \sim \pi(3+4n), \quad \Rightarrow \quad \pi(3+4n)\lambda_n = \chi(\gamma, c).$$

$3+4n$	3	7	11	15
$\chi/(\pi\lambda_n)$	3.5303	7.3547	11.1520	15.0329

Table: The values of $\chi/(\pi\lambda_n)$ for $\gamma = 5$ and $c = 0.6$.

Conclusion

Summary: in Examples 7 and 9, switching from $\lambda = 0$ to $\lambda \neq 0$ results in selecting a countable set of velocities for radiationless kink propagation.

- The first ideas about existence of such countable sets go back to the works of V.G. Gelfreich (1990,2008).

- No analytical proof of the main claim exists for now.

- The same approach can be used for homoclinic orbits (solitons)

Example 10: saturable DNLS model

$$i\psi_t + \lambda^{-2}(\psi(x+\lambda) - 2\psi(x) + \psi(x-\lambda)) + \psi(x) - \frac{\theta\psi(x)}{1+|\psi(x)|^2} = 0,$$

where θ is parameter.

- As $\lambda \to 0$, there exists the solitary wave for $\theta > 1$:

$$u''(x) + u(x) - \frac{\theta u(x)}{1+u(x)^2} = 0,$$

but it does not exist in the explicit form.

- Symmetric pairs of singularities exist for $\gamma > 0$ at $z_0 = \pm\alpha + i\beta$ with

$$u(z) = i + \sqrt{\theta}(z-z_0)\sqrt{\log(z_0 - z)}\left[1 + \mathcal{O}\left(\frac{\log|\log|z-z_0||}{\log|z-z_0|}\right)\right]$$

where the value of α can only be computed numerically and $\beta = \frac{\pi}{2\sqrt{\theta-1}}$.

Considering now the fourth-order equation

$$\varepsilon^2 u''''(x) + u''(x) + u(x) - \frac{\theta u(x)}{1 + u(x)^2} = 0,$$

we compute the Fourier integral as $k \to \infty$:

$$I(k) := \int_{\mathbb{R}} u''''(x) e^{ikx} dx = \frac{2\pi\sqrt{\theta}}{k^2\sqrt{\log k}} e^{-\beta k} \cos(\alpha k) \left[1 + \mathcal{O}\left(\frac{1}{\log k}\right)\right].$$

The infinitely many homoclinic orbits exist for

$$\varepsilon_m \sim \frac{2\alpha}{\pi(2m-1)} \quad \text{as} \quad m \to \infty.$$

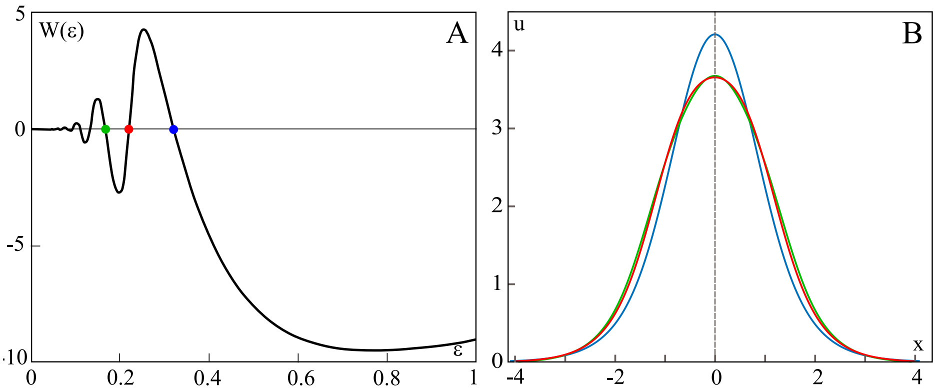


Figure: A: the plot of $W(\varepsilon) := u'''(x_P)$ versus ε for $\theta = 5$, where $u'(x_P) = 0$. Three roots exist at $\varepsilon_1 \approx 0.32$, $\varepsilon_2 \approx 0.22$ and $\varepsilon_3 \approx 0.17$. B: the profiles of soliton solutions corresponding to $\varepsilon_{1,2,3}$.

m	$\varepsilon_m = \frac{2\alpha}{\pi(2m-1)}$	Computed ε_m	$\varepsilon_m^{-1} - \varepsilon_{m-1}^{-1}$
1	0.42505	0.32128	
2	0.25503	0.22152	1.40163
3	0.18216	0.16684	1.47497
4	0.14168	0.13322	1.51259
⋮	⋮	⋮	⋮
12	0.05101	0.05029	1.55773
13	0.04723	0.04663	1.55911
14	0.04397	0.04347	1.56117

Table: The values ε corresponding to the soliton solutions at $\theta = 5$.

Conclusion

Examples 10 shows that the same mechnism is valid for homoclinic orbits, even in the case when the singularity is complicated and implicit.

Regarding the original motivation of smooth solutions of

$$\lambda^{-2}(u(x+\lambda) - 2u(x) + u(x-\lambda)) + u(x) - \frac{\theta u(x)}{1+u(x)^2} = 0,$$

we checked that the same mechanism is true for the sequence of so-called *transparent points* $\{\lambda_m\}$ (no energy difference between on-site and off-site solitons on the lattice). The spacing between $\lambda_{m+1} - \lambda_m$ is defined from the singularities in the complex plane. However, no true homoclinic orbit exist in the lattice equations because there are infinitely many resonances in the dispersion relation.

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Muchas gracias por vuestra atencion!
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