Global existence and wave breaking in the short-pulse and Ostrovsky–Hunter equations

Dmitry Pelinovsky

Department of Mathematics, McMaster University, Hamilton, Ontario, Canada
Schrödinger Institute for Mathematical Physics, Vienna, 10 June 2011

References:
Yu. Liu, D.P., A. Sakovich, Dynamics of PDE 6, 291-310 (2009)
D.P., A. Sakovich, Communications in PDE 35, 613-629 (2010)
The **Ostrovsky equation** is a model for small-amplitude long waves in a rotating fluid of a finite depth [Ostrovsky, 1978]:

\[(u_t + uu_x - \beta u_{xxx})_x = \gamma u,\]

where \(\beta\) and \(\gamma\) are real coefficients.

When \(\beta = 0\) and \(\gamma = 1\), the Ostrovsky equation is

\[(u_t + uu_x)_x = u,\]

and is known under the names of

- the short-wave equation [Hunter, 1990];
- Ostrovsky–Hunter equation [Boyd, 2005];
- reduced Ostrovsky equation [Stepanyants, 2006];
- the Vakhnenko equation [Vakhnenko & Parkes, 2002].
The **short-pulse equation** is a model for propagation of ultra-short pulses with few cycles on the pulse scale [Schäfer, Wayne 2004]:

\[ u_{xt} = u + \frac{1}{6} (u^3)_{xx}, \]

where all coefficients are normalized thanks to the scaling invariance.

The short-pulse equation

- replaces the nonlinear Schrödinger equation for short wave packets
- features exact solutions for modulated pulses
- enjoys inverse scattering and an infinite set of conserved quantities
Relevant results


- A. Stefanov et al. (2010) considered a family of the generalized short-pulse equations
  \[ u_{xt} = u + (u^p)_{xx} \]
  and proved scattering to zero for small initial data if $p \geq 4$.

- We proved both global well-posedness for small initial data and wave breaking for large initial data if $p = 3$.

- We proved wave breaking for sufficiently large initial data if $p = 2$ but found no proof of global existence for small initial data.

- C. Holliman (the group of A. Himonas) (2010-2011) proved the lack of uniform continuity with respect to initial data for a number of equations, including the Ostrovsky–Hunter and Hunter-Saxton equations,
  \[(u_t + uu_x)_x = (u_x)^2.\]
Let \( x = x(y, t) \) satisfy
\[
\begin{aligned}
x_y &= \cos w, \\
x_t &= -\frac{1}{2} w_t^2.
\end{aligned}
\]

Then, \( w = w(y, t) \) satisfies the sine–Gordon equation in characteristic coordinates [A. Sakovich, S. Sakovich, J. Phys. A 39, L361 (2006)]:
\[
w_{yt} = \sin (w).
\]

**Lemma**

Let the mapping \([0, T] \ni t \mapsto w(\cdot, t) \in H^s_c\) be \(C^1\) and
\[
H^s_c = \left\{ w \in H^s(\mathbb{R}) : \|w\|_{L^\infty} \leq w_c < \frac{\pi}{2} \right\}, \quad s \geq 1.
\]

Then, \( x(y, t) \) is invertible in \( y \) for any \( t \in [0, T] \) and \( u(x, t) = w_t(y(x, t), t) \)
solves the short-pulse equation
\[
 u_{xt} = u + \frac{1}{6} (u^3)_{xx}, \quad x \in \mathbb{R}, \quad t \in [0, T].
\]
A kink of the sine–Gordon equation gives a loop solution of the short-pulse equation:

\[
\begin{cases}
  u = 2 \text{sech}(y + t), \\
  x = y - 2 \tanh(y + t).
\end{cases}
\]

**Figure:** The loop solution \( u(x, t) \) to the short-pulse equation
Solutions of the short-pulse equation

A breather of the sine–Gordon equation gives a pulse solution of the short-pulse equation:

\[
\begin{aligned}
    u(y, t) &= 4mn \frac{m \sin \psi \sinh \phi + n \cos \psi \cosh \phi}{m^2 \sin^2 \psi + n^2 \cosh^2 \phi} = u\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right), \\
    x(y, t) &= y + 2mn \frac{m \sin 2\psi - n \sinh 2\phi}{m^2 \sin^2 \psi + n^2 \cosh^2 \phi} = x\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right) + \frac{\pi}{m},
\end{aligned}
\]

where

\[
\begin{align*}
    \phi &= m(y + t), \\
    \psi &= n(y - t), \\
    n &= \sqrt{1 - m^2},
\end{align*}
\]

and \(m \in \mathbb{R}\) is a free parameter.

Figure: The pulse solution to the short-pulse equation with \(m = 0.25\)
Local well-posedness of the short-pulse equation

**Theorem (Schäfer & Wayne, 2004)**

Let $u_0 \in H^2$. There exists a maximal existence time $T = T(u_0) > 0$ and a unique solution to the short-pulse equation

$$u(t) \in C([0, T), H^2) \cap C^1([0, T), H^1)$$

that satisfies $u(0) = u_0$ and depends continuously on $u_0$.

**Remarks:**

- The proof can be extended to any $s > \frac{3}{2}$ (Stefanov et al, 2010).
- There is a constraint on solutions of the short-pulse equation

$$\int_{\mathbb{R}} u(x, t) dx = 0, \quad t > 0.$$
Consider the Cauchy problem for the sine-Gordon equation

$$\begin{cases} w_{yt} = \sin w, & y \in \mathbb{R}, \ t > 0 \\ w|_{t=0} = w_0, & y \in \mathbb{R}. \end{cases}$$

**Note:** if $w \in C^1([0, T), H^s(\mathbb{R}))$, $s > \frac{1}{2}$, then

$$\int_{\mathbb{R}} \sin w(y, t) dy = 0, \ t \in (0, T).$$

The standard method of Picard–Kato would not work because if $w(\cdot, t) \in H^s$, $s > \frac{1}{2}$, then $\sin(w(\cdot, t)) \in H^s$, but $\partial_y^{-1} \sin(w(y, t)) dy$ may not be in $H^s$.

Let $q = \sin(w)$ and rewrite the Cauchy problem in the equivalent form

$$\begin{cases} q_t = (1 - f(q)) \partial_y^{-1} q, \\ q|_{t=0} = q_0, \end{cases}$$

where

$$f(q) := 1 - \sqrt{1 - q^2} = \frac{q^2}{1 + \sqrt{1 - q^2}}, \ \forall |q| \leq 1: \ \frac{q^2}{2} \leq f(q) \leq q^2.$$
Consider the initial-value problem

\[
\begin{cases}
q_t = (1 - f(q))\partial_y^{-1} q, \\
q|_{t=0} = q_0.
\end{cases}
\]

Now the constraints are

\[\|q(\cdot, t)\|_{L^\infty} < 1, \quad \int_\mathbb{R} q(y, t) dy = 0, \quad t > 0.\]

**Theorem**

Assume that \(q_0 \in X^s_c, s > \frac{1}{2},\) where

\[X^s_c = \left\{ q \in H^s \cap \dot{H}^{-1}, \, \|q\|_{L^\infty} \leq q_c < 1 \right\} .\]

There exist a maximal time \(T = T(q_0) > 0\) and a unique solution \(q(t) \in C([0, T), X^s_c)\) of the Cauchy problem that satisfies \(q(0) = q_0\) and depends continuously on \(q_0.\)
Consider the Cauchy problem for the linearized sine–Gordon equation

\[
\begin{align*}
Q_t &= \partial_y^{-1} Q, \\
Q|_{t=0} &= Q_0.
\end{align*}
\]

Denote

\[ L = \partial_y^{-1} \quad \text{and} \quad Q(t) = e^{tL} Q_0. \]

The solution operator \( e^{tL} \) is an \textit{isometry} from \( H^s \) to \( H^s \) for any \( s \geq 0 \), so that

\[ \| Q(t) \|_{H^s} = \| e^{tL} Q_0 \|_{H^s} = \| Q_0 \|_{H^s}, \quad \forall t \in \mathbb{R}. \]

By Duhamel’s principle, we have

\[ q(t) = e^{tL} q_0 - \int_0^t e^{(t-t')L} f(q(t')) \partial_y^{-1} q \, dt'. \]
Fix $q_c \in (0, 1)$, $\delta > 0$ and $\alpha \in (0, 1)$ so that the initial data satisfy

$$\|q_0\|_{X^s} \leq \alpha \delta, \quad \|q_0\|_{L^\infty} \leq \alpha q_c$$

We need to show that there exists $T > 0$ such that

- the mapping

$$ (Aq)(t) = \int_0^t e^{(t-t')L} f(q(t')) \partial_y^{-1} q(t') dt' : C([0, T], X^s_c) \mapsto C([0, T], X^s_c) $$

is Lipschitz continuous and a contraction for sufficiently small $T > 0$.

- The integral equation is well-defined in

$$\|q(t)\|_{X^s} \leq \delta, \quad \|q(t)\|_{L^\infty} \leq q_c, \quad t \in [0, T].$$

Existence, uniqueness, and continuous dependence come from the standard Banach’s Fixed-Point Theorem.
Our local well-posedness of the short-pulse equation

**Theorem (P., Sakovich, 2010)**

Let $u_0 \in H^s \cap \dot{H}^{-1}$, $s > 3/2$. There exists a maximal existence time $T = T(u_0) > 0$ and a unique solution to the short-pulse equation

$$u(t) \in C^1([0, T), H^s \cap \dot{H}^{-1})$$

that satisfies $u(0) = u_0$ and depends continuously on $u_0$.

This theorem follows from the local well-posedness of the sine–Gordon equation and the correspondence

$$u = w_t = \frac{q_t}{\sqrt{1 - q^2}} = p, \quad u_x = \frac{w_{ty}}{\cos(w)} = \tan(w) = \frac{p_y}{\sqrt{1 - q^2}}.$$
A bi-infinite hierarchy of conserved quantities of the short-pulse equation was found in Brunelli [J.Math.Phys. 46, 123507 (2005)]:

\[ E_{-1} = \int_{\mathbb{R}} \left( \frac{1}{24} u^4 - \frac{1}{2} (\partial_x^{-1} u)^2 \right) dx, \]
\[ E_0 = \int_{\mathbb{R}} u^2 dx, \]
\[ E_1 = \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} dx, \]
\[ E_2 = \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} dx, \]
\[ \ldots \]
Global well-posedness of the short-pulse equation

**Theorem (P. & Sakovich, 2010)**

Let $u_0 \in H^2$ and the conserved quantities satisfy $2E_1 + E_2 < 1$. Then the short-pulse equation admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ with $u(0) = u_0$.

The values of $E_0$, $E_1$ and $E_2$ are bounded by $\|u_0\|_{H^2}$ as follows:

\[
E_0 = \int_{\mathbb{R}} u^2 \, dx = \|u_0\|_{L^2}^2,
\]
\[
E_1 = \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} \, dx \leq \frac{1}{2} \|u'_0\|_{L^2}^2,
\]
\[
E_2 = \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} \, dx \leq \|u''_0\|_{L^2}^2.
\]

The existence time $T > 0$ of the local solutions is inverse proportional to the norm $\|u_0\|_{H^2}$ of the initial data. To extend $T$ to $\infty$, we need to control the norm $\|u(t)\|_{H^2}$ by a $T$-independent constant on $[0, T]$. 
Let \( \tilde{q}(x, t) = \frac{u_x}{\sqrt{1+u_x^2}} \). Then, we obtain

\[
\|\tilde{q}(t)\|_{H^1} \leq \sqrt{2E_1 + E_2} < 1, \quad t \in [0, T).
\]

Thanks to Sobolev's embedding \( \|\tilde{q}\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|\tilde{q}\|_{H^1} < 1, \) so that \( u_x = \frac{\tilde{q}}{\sqrt{1-\tilde{q}^2}} \) satisfies the bound

\[
\|u_x(t)\|_{H^1} \leq \frac{\|\tilde{q}\|_{H^1}}{\sqrt{1 - \|\tilde{q}\|_{H^1}^2}}, \quad t \in [0, T)
\]

or equivalently

\[
\|u(t)\|_{H^2} \leq \left( E_0 + \frac{2E_1 + E_2}{1 - (2E_1 + E_2)} \right)^{1/2}, \quad t \in [0, T).
\]
Corollary

Let \( u_0 \in H^2 \) such that \( 2\sqrt{2E_1 E_2} < 1 \). Then the short-pulse equation admits a unique solution \( u(t) \in C(\mathbb{R}_+, H^2) \) with \( u(0) = u_0 \).

Let \( \alpha \in \mathbb{R}_+ \) be an arbitrary parameter. If \( u(x, t) \) is a solution of the short-pulse equation, then \( U(X, T) \) is also a solution with

\[
X = \alpha x, \quad T = \alpha^{-1} t, \quad U(X, T) = \alpha u(x, t).
\]

The scaling invariance yields transformation \( \tilde{E}_1 = \alpha E_1 \) and \( \tilde{E}_2 = \alpha^{-1} E_2 \). For a given \( u_0 \in H^2 \), a family of initial data \( U_0 \in H^2 \) satisfies

\[
\phi(\alpha) = 2\tilde{E}_1 + \tilde{E}_2 = 2\alpha E_1 + \alpha^{-1} E_2 \geq 2\sqrt{2E_1 E_2}, \quad \forall \alpha \in \mathbb{R}_+.
\]

If \( 2\sqrt{2E_1 E_2} < 1 \), there exists \( \alpha \) such that \( U(X, T) \) is defined for any \( T \in \mathbb{R}_+ \).
Short-pulse equation in a periodic domain

Let $\mathbb{S}$ be the unit circle and let $\partial_x^{-1}$ be the mean-zero anti-derivative

$$
\partial_x^{-1} u = \int_0^x u(x', t) \, dx' - \int_{\mathbb{S}} \int_0^x u(x', t) \, dx' \, dx.
$$

The short-pulse equation on a circle is given by

$$
\begin{aligned}
\frac{u}{t} &= \frac{1}{2} u^2 u_x + \partial_x^{-1} u, \\
u(x, 0) &= u_0(x), & x \in \mathbb{S}, & t \geq 0.
\end{aligned}
$$

Let $u(t) \in C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S}))$ be a local solution such that $u(0) = u_0 \in H^s(\mathbb{S})$.

- The assumption $\int_{\mathbb{S}} u_0(x) \, dx = 0$ is necessary for existence.
- The following quantities are constant on $[0, T)$:

$$
E_0 = \int_{\mathbb{S}} u^2 \, dx, \quad E_1 = \int_{\mathbb{S}} \sqrt{1 + u_x^2} \, dx
$$
Lemma

Let $u_0 \in H^2(S)$ and $u(t)$ be a local solution of the Cauchy problem. The solution blows up in a finite time $T < \infty$ in the sense $\lim_{t \uparrow T} \|u(\cdot, t)\|_{H^2} = \infty$ if and only if

$$\limsup_{t \uparrow T} \sup_{x \in S} u(x, t)u_x(x, t) = +\infty.$$ 

For the inviscid Burgers equation

$$\begin{cases}
u_t = \frac{1}{2} u^2 u_x, \\ u(x, 0) = u_0(x),
\end{cases}$$

$x \in S, \ t \geq 0.$

the problem can be solved by the method of characteristics. The finite-time blow-up occurs for any $u_0(x) \in C^1(S)$ if there is a point $x_0 \in S$ such that $u_0(x_0)u_0'(x_0) > 0$. The blow-up time is

$$T = \inf_{\xi \in S} \left\{ \frac{1}{u_0(\xi)u_0'(\xi)} : \ u_0(\xi)u_0'(\xi) > 0 \right\}.$$
Let $\xi \in S$, $t \in [0, T)$, and denote

$$x = X(\xi, t), \quad u(x, t) = U(\xi, t), \quad \partial_x^{-1} u(x, t) = G(\xi, t).$$

At characteristics $x = X(\xi, t)$, we obtain

$$\begin{align*}
\left\{ \begin{array}{l}
\dot{X}(t) = -\frac{1}{2} U^2, \\
X(0) = \xi,
\end{array} \right. \quad \left\{ \begin{array}{l}
\dot{U}(t) = G, \\
U(0) = u_0(\xi),
\end{array} \right.
\end{align*}$$

- The map $X(\cdot, t) : S \mapsto \mathbb{R}$ is an increasing diffeomorphism with

$$\partial_\xi X(\xi, t) = \exp \left( \int_0^t u(X(\xi, s), s)u_x(X(\xi, s), s)ds \right) > 0, \quad t \in [0, T), \quad \xi \in S.$$

- The following quantities are bounded on $[0, T)$:

$$|u(x, t)| \leq \left| \int_{\xi_t}^x u_x(x, t) \, dx \right| \leq \int_S |u_x(x, t)| \, dx \leq E_1$$

and

$$|\partial_x^{-1} u(x, t)| \leq \left| \int_{\xi_t}^x u(x, t) \, dx \right| \leq \int_S |u(x, t)| \, dx \leq \sqrt{E_0}.$$
Theorem (Liu, P. & Sakovich, 2009)

Let \( u_0 \in H^2(S) \) and \( \int_S u_0(x) \, dx = 0 \). Assume that there exists \( x_0 \in \mathbb{R} \) such that \( u_0(x_0)u'_0(x_0) > 0 \) and

\[
\text{either } |u'_0(x_0)| > \left( \frac{E_1^2}{4E_0^{1/2}} \right)^{1/3},
\]

\[
|u_0(x_0)||u'_0(x_0)|^2 > E_1 + \left( 2E_0^{1/2}|u'_0(x_0)|^3 - \frac{1}{2}E_1^2 \right)^{1/2},
\]

or \( |u'_0(x_0)| \leq \left( \frac{E_1^2}{4E_0^{1/2}} \right)^{1/3}, \quad |u_0(x_0)||u'_0(x_0)|^2 > E_1. \)

Then there exists a finite time \( T \in (0, \infty) \) such that the solution \( u(t) \in C([0, T), H^2(S)) \) of the Cauchy problem blows up with the property

\[
\limsup_{t \uparrow T} \sup_{x \in S} u(x, t)u_x(x, t) = +\infty, \quad \text{while} \quad \lim_{t \uparrow T} \|u(\cdot, t)\|_{L^\infty} \leq E_1.
\]
Sketch of the proof

Let $V(\xi, t) = u_x(X(\xi, t), t)$ and $W(\xi, t) = U(\xi, t)V(\xi, t)$. Then

$$\begin{align*}
\dot{V} &= VW + U,
\dot{W} &= W^2 + VG + U^2.
\end{align*}$$

Under the conditions of the theorem, there exists $\xi_0 \in S$ such that $V(\xi_0, t)$ and $W(\xi_0, t)$ satisfy the apriori estimates

$$\begin{align*}
\dot{V} &\geq VW - E_1,
\dot{W} &\geq W^2 - V\sqrt{E_0}.
\end{align*}$$

We show that $V(\xi_0, t)$ and $W(\xi_0, t)$ go to infinity in a finite time.
Consider Gaussian initial data

\[ u_0(x) = a(1 - 2bx^2)e^{-bx^2}, \quad x \in \mathbb{R}, \]

where \((a, b)\) are arbitrary and \(\int_{\mathbb{R}} u_0(x)dx = 0\) is satisfied.

**Figure:** Global solutions exist in the red region and wave breaking occurs in the blue region.
Using the pseudospectral method, we solve

\[ \frac{\partial}{\partial t} \hat{u}_k = -i \frac{k}{k} \hat{u}_k + \frac{ik}{6} \mathcal{F} \left[ (\mathcal{F}^{-1} \hat{u})^3 \right]_k, \quad k \neq 0, \quad t > 0. \]

Consider the 1-periodic initial data

\[ u_0(x) = a \cos(2\pi x) \]

- Criterion for wave breaking: \( a > 1.053 \).
- Criterion for global solutions: \( a < 0.0354 \).
Evolution of the cosine initial data

Figure: Solution surface $u(x, t)$ (left) and the supremum norm $W(t)$ (right) for $a = 0.2$ (top) and $a = 0.5$ (bottom).
We compute the best power fit for
\[ W(t) := \sup_{x \in S} u(x, t) u_x(x, t) \]
according to the blow-up law
\[ W(t) \approx \frac{C}{T - t} \quad \text{for} \quad 0 < T - t \ll 1. \]

Note that the inviscid Burgers equation has the exact blow-up law
\[ W(t) = \frac{1}{T - t}. \]

**Figure:** Time of wave breaking $T$ versus $a$ (left). Constant $C$ of the linear regression versus $a$ (right).
The Ostrovsky–Hunter equation

Cauchy problem on a circle $\mathbb{S}$ of unit length:

$$\begin{cases} 
    u_t + uu_x = \partial_x^{-1} u, & t > 0, \\
    u(0, x) = u_0(x), 
\end{cases}$$

where

$$\partial_x^{-1} u := \int_0^x u(t, x') dx' - \int_\mathbb{S} \int_0^x u(t, x') dx' dx.$$ 

The inviscid Burgers equation $u_t + uu_x = 0$ develops wave breaking in a finite time for any initial data $u(0, x) = u_0(x)$ if $u_0(x) \in C^1$ and there is a point $x_0$ such that $u'(x_0) < 0$. In other words, there exists a finite time $T \in (0, \infty)$ such that

$$\lim \inf_{t \uparrow T} \inf_x u_x(t, x) = -\infty, \quad \text{while} \quad \lim \sup_{t \uparrow T} \sup_x |u(t, x)| < \infty.$$ 

Moreover, the blow-up time is computed by the method of characteristics:

$$T = \inf_\xi \left\{ \frac{1}{|u'(\xi)|} : u_0'(\xi) < 0 \right\}.$$
For the Ostrovsky–Hunter equation, it was found that

**Theorem (Hunter, 1990)**

Let \( u_0(x) \in C^1(S) \), where \( S \) is a circle of unit length, and define

\[
\inf_{x \in S} u'_0(x) = -m \quad \text{and} \quad \sup_{x \in S} |u_0(x)| = M.
\]

If \( m^3 > 4M(4 + m) \), a smooth solution \( u(t, x) \) breaks down at a finite time.

Note that there exist infinitely many conserved quantities of the Ostrovsky-Hunter equations, which are not useful:

\[
E_0 = \int_{\mathbb{R}} u^2 \, dx,
\]

\[
E_{-1} = \int_{\mathbb{R}} \left( \frac{1}{3} u^3 + (\partial_x^{-1} u)^2 \right) \, dx,
\]

\[
\ldots
\]
Let \( u_0(x) \in H^s(S), \ s > \frac{3}{2} \) and \( u(t, x) \) be a solution of the Cauchy problem. The solution blows up in a finite time \( T \in (0, \infty) \) in the sense of \( \lim_{t \uparrow T} \| u(t, \cdot) \|_{H^s} = \infty \) if and only if

\[
\lim \inf_{t \uparrow T, x \in S} u_x(t, x) = -\infty \quad \text{while} \quad \lim \sup_{t \uparrow T, x} |u(t, x)| < \infty.
\]

We have

\[
|\partial_x^{-1} u(t, x)| \leq \int_S |u(t, x)| \, dx \leq \| u \|_{L^2} = \| u \|_{L^2}
\]

and

\[
\sup_{s \in [0, t]} \| u(s, \cdot) \|_{L^\infty} \leq \| u_0 \|_{L^\infty} + t \| u_0 \|_{L^2}, \quad \forall t \in [0, T).
\]
Sufficient condition for wave breaking

**Theorem**

Assume that $u_0(x) \in H^s(S)$, $s > \frac{3}{2}$ and $\int_S u_0(x) \, dx = 0$. If either

$$\int_S (u'_0(x))^3 \, dx < - \left( \frac{3}{2} \| u_0 \|_{L^2} \right)^{3/2},$$

(1)

or

$$\int_S (u'_0(x))^3 \, dx < 0 \quad \text{and} \quad \| u_0 \|_{L^2} > \frac{3}{4},$$

(2)

or there is a $x_0 \in S$ such that

$$u'_0(x_0) \leq -(1 + \epsilon) \left( \| u_0 \|_{L^\infty} + T_1 \| u_0 \|_{L^2} \right)^{1/2},$$

(3)

where $T_1$ is the smallest positive root of

$$2T_1 \left( \| u_0 \|_{L^\infty} + T_1 \| u_0 \|_{L^2} \right)^{1/2} = \log \left( 1 + \frac{2}{\epsilon} \right),$$

then the solution $u(t, x)$ of the Cauchy problem blows up in a finite time.
Proof of sufficient conditions (1)–(2)

Direct computation gives

\[
\frac{d}{dt} \int_S u_x^3 \, dx = 3 \int_S u_x^2 (-u_x^2 - uu_{xx} + u) \, dx
\]

\[
= -2 \int_S u_x^4 \, dx + 3 \int_S uu_x^2 \, dx
\]

\[
\leq -2 \left\| u_x \right\|^4_{L^4} + 3 \left\| u \right\|_{L^2} \left\| u_x \right\|^2_{L^4}.
\]

By Hölder’s inequality, we have

\[
|V(t)| \leq \left\| u_x \right\|^3_{L^3} \leq \left\| u_x \right\|^3_{L^4}, \quad V(t) = \int_S u_x^3(t, x) \, dx < 0.
\]

Let \( Q_0 = \left\| u \right\|^2_{L^2} = \left\| u_0 \right\|^2_{L^2} \) and \( V(0) < -\left( \frac{3}{2} Q_0 \right)^{\frac{3}{2}} \). Then,

\[
\frac{dV}{dt} \leq -2 \left( \left| V \right|^\frac{2}{3} - \frac{3Q_0}{4} \right)^2 + \frac{9Q_0^2}{8},
\]

There is \( T < \infty \) such that \( V(t) \to -\infty \) as \( t \uparrow T \).
Proof of sufficient condition (3)

Let $\xi \in S$, $t \in [0, T)$, and denote

$$x = X(\xi, t), \quad u(x, t) = U(\xi, t), \quad \partial_x^{-1}u(x, t) = G(\xi, t).$$

At characteristics $x = X(\xi, t)$, we obtain

$$\begin{cases}
\dot{X}(t) = U, \\
X(0) = \xi,
\end{cases} \quad \begin{cases}
\dot{U}(t) = G, \\
U(0) = u_0(\xi),
\end{cases}$$

Let $V(\xi, t) = u_x(t, X(\xi, t))$. Then

$$\dot{V} = -V^2 + U \quad \Rightarrow \quad \dot{V} \leq -V^2 + (\|u_0\|_{L^\infty} + \gamma t\|u_0\|_{L^2})$$
Using the pseudospectral method, we solve

\[ \frac{\partial}{\partial t} \hat{u}_k = -\frac{i}{k} \hat{u}_k - \frac{ik}{2} \mathcal{F} \left[ (\mathcal{F}^{-1} \hat{u})^2 \right]_k, \quad k \neq 0, \quad t > 0. \]

Consider the 1-periodic initial data

\[ u_0(x) = a \cos(2\pi x) + b \sin(4\pi x), \]
Evolution of the cosine initial data

Figure: Solution surface $u(t, x)$ (left) and $\inf_{x \in S} u_x(t, x)$ versus $t$ (right) for $a = 0.005$, $b = 0$ (top) and $a = 0.05$, $b = 0$ (bottom). $C \approx -1.009$ and $B \approx 3.213$. 
Summary

- We found sufficient conditions for global well-posedness of the short-pulse equation for small initial data.

- We found sufficient conditions for wave breaking of the short-pulse and Ostrovsky–Hunter equations for large initial data.

- We illustrated both global existence and wave breaking numerically.

- Numerical results suggest orbital stability of the exact modulated pulses of the short-pulse equation.

- Numerical results suggest global existence for small initial data in the Ostrovsky-Hunter equation.