Instability of the peaked traveling wave in a local model for shallow water waves

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Outline of the talk

1. The model for shallow water waves and motivations.
2. Conserved quantities and local well-posedness.
3. Coexistence of smooth, peaked, cusped traveling waves.
4. Instability of the peaked traveling wave.
5. Convergence to the peaked wave along the family of smooth waves.

The talk is based on collaborations with

- Spencer Locke (PhD at University of Michigan, USA),
- Shuoyang Wang (BSc at McMaster University, Canada),
- Fabio Natali (University of Maringa, Brazil),
- Sergey Dyachenko (University of Buffalo, USA).

1. The model and motivations.

We consider the following PDE for $\eta = \eta(t,x) : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$,

$$\boxed{2c\partial_x\partial_t\eta = (c^2 - 2\eta)\partial_x^2\eta - (\partial_x\eta)^2 + \eta}$$

with x defined in the 2π-periodic domain $\mathbb{T}$ and $c > 0$ being a parameter for the wave speed.

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with x defined in the 2π-periodic domain $\mathbb{T}$ and $c > 0$ being a parameter for the wave speed.

- High-frequency limit of the Camassa–Holm equation

$$u_T - u_{TXX} + ku_X + 3uu_X = 2u_X u_{XX} + uu_{XXX},$$

for solutions of the form $u(T,X) = 2\eta(t,x)$ with $t = 2c\varepsilon^{-1}T$, $x = \varepsilon^{-1}(X - c^2T)$, $k = \varepsilon^{-2}$ in the formal limit $\varepsilon \to 0$.

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with x defined in the 2π-periodic domain $\mathbb{T}$ and $c > 0$ being a parameter for the wave speed.

- Extension of the Hunter–Saxton equation

$$(u_T + uu_X)_X = \frac{1}{2}(u_X)^2$$

for solutions of the form $u(T,X) = 2\eta(t,x)$ with $t = 2cT$, $x = X + c^2T$, without the last term.

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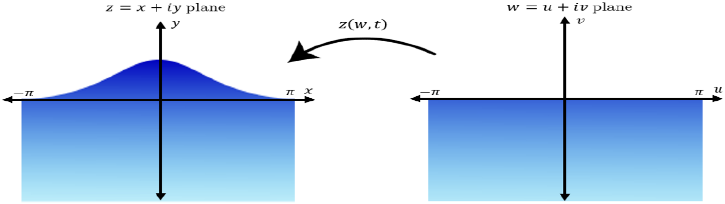
with x defined in the 2π-periodic domain $\mathbb{T}$ and $c > 0$ being a parameter for the wave speed.

- Truncation of Euler's equation in conformal variables with η being the surface elevation and $c > 0$ being wave speed.



S.Locke–D.P., JFM 2025

Traveling waves with smooth and peaked profiles

Global bifurcation results suggest that $\exists$ ∞-many bifurcations of Stokes waves with smooth profiles when the steepness is increased towards the limiting wave with the peaked profile.

B. Buffoni E. N. Dancer, J. F. Toland, ARMA 152 (2000) 241

Existence of smooth Stokes waves of large amplitudes was obtained numerically with a high precision.

S. Dyachenko, P. Lushnikov, A. Korotkevich, SAPM 137 (2016) 419

S. Dyachenko, V. Hur, D. Silantyev, JFM 955 (2023) A17

Spectral stability of smooth Stokes waves of large amplitudes was explored numerically for perturbations with same period.

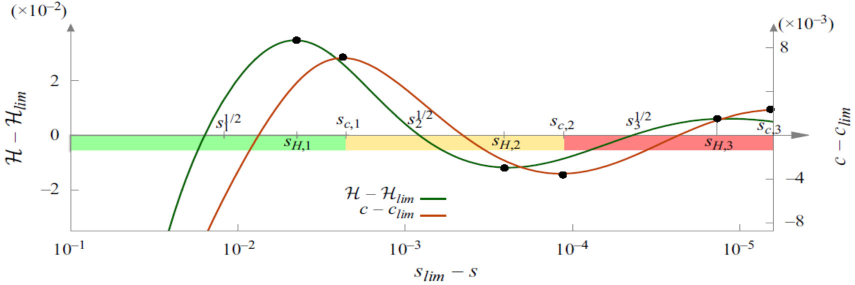
S. Dyachenko, A. Semenova, JCP 492 (2023) 112411

A. Korotkevich, P. Lushnikov, A. Semenova, S. Dyachenko, SAPM 150 (2023)

B. Deconinck, S. Dyachenko, A. Semenova, JFM 995 (2024) A2

Traveling waves with smooth and peaked profiles

The wave energy H and the wave speed c oscillate as functions of the wave steepness s towards the limiting peaked wave:



- $\exists$ ∞-many oscillations of wave energy and speed.
- At each extremum of speed, the Morse index is increased by one. It is infinite at the limiting peaked wave.
- At each extremum of energy, a new unstable eigenvalue bifurcates in the stability problem.

Traveling waves with smooth and peaked profiles

The traveling wave of the peaked profile is believed to exist for a unique value of $c = c_*$. However, the proof of uniqueness of c_* and the convergence as $c \to c_*$ for Stokes waves is open.

For model equations (fractional KdV, Whitham), the proof of uniqueness of c_* is developed in

A. Geyer & D.P, SIMA, 2019

G. Bruell & Dhara, Indiana Math. J. 2021

J. Dahne, J. Diff. Eqs. 401 (2024) 550

M. Ehrnström, O.I.H. Mæhlen, K. Varholm, Ann. Inst. H. Poincaré C (2025)

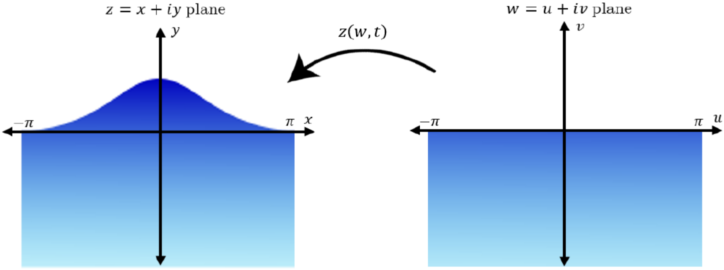
Babenko's equation for traveling (Stokes) waves

Traveling waves $\eta(u,t) = \eta(u - ct)$ with the zero-mean profile η satisfy a scalar pseudo–differential equation

$$\boxed{(c^2 \mathcal{K}_h - 1)\eta = \frac{1}{2}\mathcal{K}_h \eta^2 + \eta \mathcal{K}_h \eta,}$$

where the self-adjoint operator $\mathcal{K}_h$ is defined by

$$\widehat{(\mathcal{K}_h)}_n = \begin{cases} n \coth(hn), & n \in \mathbb{Z}\backslash\{0\}, \\ 0, & n = 0. \end{cases}$$



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Existence of traveling waves with both smooth and peaked profiles is defined by solutions of Babenko's equation.

K. Babenko, Russian Academy of Sciences 294 (1987) 1033

⋮

S. Locke & D.P., Appl. Math. Lett. 161 (2025) 109359

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The zero-mean constraint on η in the physical coordinate becomes the quadratic constraint in the conformal coordinate:

$$\oint \eta + \oint \eta\mathcal{K}_h\eta = 0.$$

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In the deep-water limit $h \to \infty$, we have $\mathcal{K}_h \to |\partial_u|$ and Babenko's equation becomes the "stationary BO" equation

$$(c^2|\partial_u| - 1)\eta = \frac{1}{2}|\partial_u|\eta^2 + \eta|\partial_u|\eta,$$

with ∞-many oscillations for c in $(1, c_{\max})$ with $c_* \approx 1.0922$.

Babenko's equation for traveling (Stokes) waves

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where the self-adjoint operator $\mathcal{K}_h$ is defined by

$$\widehat{(\mathcal{K}_h)}_n = \begin{cases} n \coth(hn), & n \in \mathbb{Z}\backslash\{0\}, \\ 0, & n = 0. \end{cases}$$

In the shallow-water limit $h \to 0$, we replace $\mathcal{K}_h$ by $-\partial_u^2$ and Babenko's equation becomes the stationary equation

$$(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0,$$

which is equivalent to the steady model considered here.

2. Conserved quantities and local well-posedness.

Assume a smooth solution $\eta \in C^0((-\tau_0, \tau_0), H^s_{\mathrm{per}}(\mathbb{T}))$, $s > \frac{3}{2}$:

$$\boxed{2c\partial_x\partial_t\eta = (c^2 - 2\eta)\partial_x^2\eta - (\partial_x\eta)^2 + \eta}$$

The model has the three (basic) conserved quantities:

- Mass

$$M(\eta) = \oint \eta dx$$

- Momentum

$$Q(\eta) = \frac{1}{2}\oint (\partial_x\eta)^2 dx$$

- Energy

$$H(\eta) = \frac{1}{2}\oint \left[\eta^2 + 2\eta(\partial_x\eta)^2\right] dx.$$

Moreover, it admits a nontrivial constraint

$$M(\eta) + 2Q(\eta) = \oint \left[\eta + (\partial_x\eta)^2\right] dx = 0.$$

Local well-posedness of the initial-value problem

Integrating once in x,

$$2c\partial_x\partial_t\eta = (c^2 - 2\eta)\partial_x^2\eta - (\partial_x\eta)^2 + \eta$$

we can write the evolution problem

$$\boxed{2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta + \Pi_0\partial_x^{-1}\Pi_0\left[(\partial_x\eta)^2 + \eta\right]}$$

where $\Pi_0 : L^2(\mathbb{T}) \to L^2(\mathbb{T})|_{\{1\}^\perp}$.

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The mapping

$$\Pi_0\partial_x^{-1}\Pi_0\left[(\partial_x\eta)^2 + \eta\right] : H^1_{\mathrm{per}} \cap W^{1,\infty} \to H^1_{\mathrm{per}} \cap W^{1,\infty}$$

is bounded on every bounded subset.

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The inviscid Burgers equation

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta$$

is locally well-posed in $H^1_{\mathrm{per}} \cap W^{1,\infty}$
(e.g., the method of characteristics).

Local well-posedness of the initial-value problem

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- The initial-value problem is locally well-posed in $H^1_{\mathrm{per}} \cap W^{1,\infty}$. For CH, it was known from
  C. DeLellis, T. Kappeler, P. Topalov, Comm. PDEs 32 (2007) 87
  F. Linares, G. Ponce, T. C. Sideris (2019)

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where $\Pi_0 : L^2(\mathbb{T}) \to L^2(\mathbb{T})|_{\{1\}^\perp}$.

- The initial-value problem for the evolution probem is also locally well-posed in smoother Sobolev space $H^s_{\rm per}$, $s > \frac{3}{2}$, which is embedded into $H^1_{\rm per} \cap W^{1,\infty}$.
  W. Ye, Z. Yin, Monats. Math. 191 (2020) 267

Local well-posedness of the initial-value problem

Integrating once in x,

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- IVP is ill-posed in H^s_{per}, $s \leq \frac{3}{2}$.
  A. Himonas, K. Grayshan, C. Holliman, J. Nonlinear Sci. 26 (2016) 1175
  Z. Guo, X. Liu, L. Molinet, Z. Yin, J. Diff. Eqs. 266 (2019) 1698

3. Smooth and peaked traveling waves.

Traveling waves are defined by solutions of the 2^{nd}-order ODE:

$$\boxed{(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0, \qquad x \in \mathbb{T}.}$$

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$$\boxed{(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0, \qquad x \in \mathbb{T}.}$$

Theorem (S. Locke–D. P., JFM, 2025)

There exist $c_* := \frac{\pi}{2\sqrt{2}}$ and $c_\infty \in (c_*, \infty)$ such that the ODE admits a unique solution with the profile $\eta \in C^\infty_{\text{per}}(\mathbb{T})$ for every $c \in (1, c_*)$ s.t.

$$\|\eta\|_{L^\infty} \to 0 \quad \text{as} \quad c \to 1$$

and a solution with the profile $\eta \in C^0_{\text{per}}(\mathbb{T})$ for every $c \in (c_*, c_\infty)$ satisfying for some $A(c) > 0$,

$$\eta(x) = \frac{c^2}{2} - A(c)|x|^{2/3} + \mathcal{O}(|x|) \quad \text{as} \quad x \to 0.$$

3. Smooth and peaked traveling waves.

Traveling waves are defined by solutions of the $2^{\rm nd}$-order ODE:

$$\boxed{(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0, \qquad x \in \mathbb{T}.}$$

- The two continuous families meet at $c = c_* = \frac{\pi}{2\sqrt{2}}$, where the profile $\eta \in C^0_{\rm per}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ is peaked:

$$\eta(x) = \frac{1}{16}(\pi^2 - 4\pi|x| + 2x^2), \qquad x \in [-\pi, \pi].$$

  with the highest amplitude from Bernoulli's principle of hydrodynamics: $\max_{x \in \mathbb{T}} \eta(x) = \eta(0) = \frac{c^2}{2}$

- The $|x|^{2/3}$ singularity in the conformal coordinate (cusped waves) corresponds to Stokes' law of the 120^0 angle in the physical coordinate.

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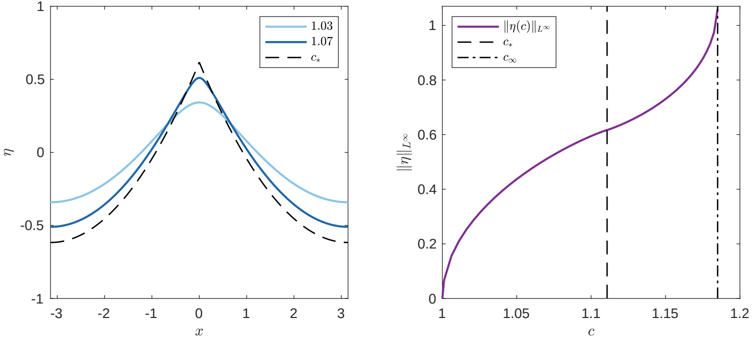

Existence

Smooth solutions of $\boxed{(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0}$ are level curves of

$$E(\eta, \eta') := \frac{1}{2}(c^2 - 2\eta)(\eta')^2 + \frac{1}{2}\eta^2 = \mathcal{E}$$

on the phase plane (η, η').

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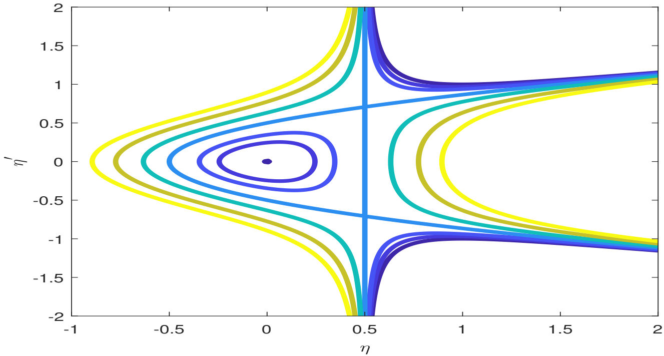

Stability

The evolution equation

$$\boxed{2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta + \Pi_0\partial_x^{-1}\Pi_0\left[(\partial_x\eta)^2 + \eta\right]}$$

has the Hamiltonian formulation

$$2c\partial_t\eta = J\nabla\left[H(\eta) - c^2Q(\eta)\right], \quad J := \Pi_0\partial_x^{-1}\Pi_0.$$

Here H is the energy and Q is the momentum given by

$$Q(\eta) = \frac{1}{2}\oint(\partial_x\eta)^2dx, \quad H(\eta) = \frac{1}{2}\oint\left[\eta^2 + 2\eta(\partial_x\eta)^2\right]dx.$$

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The profile $\eta \in C^\infty_{\mathrm{per}}(\mathbb{T})$ for smooth traveling waves is a critical point of $H - c^2Q$. The Hessian operator at η is the Sturm–Liouville operator $\mathcal{L} : H^2_{\mathrm{per}} \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$,

$$\mathcal{L} = -\partial_x(c^2 - 2\eta)\partial_x + (2\eta'' - 1).$$

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$$\mathcal{L} = -\partial_x(c^2 - 2\eta)\partial_x + (2\eta'' - 1).$$

The spectrum of $\mathcal{L} : H^2_{\rm per}(\mathbb{T}) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is purely discrete. The coefficients of $\mathcal{L}$ are singular in the limit of peaked wave since $2\eta''(x) - 1 \to -\frac{1}{2} - \pi\delta_0$ with Dirac δ_0.

Linear stability of smooth waves

The linear evolution $2c\partial_t \zeta = -J\mathcal{L}\zeta$ is defined by

$$\boxed{J = \Pi_0 \partial_x^{-1} \Pi_0, \quad \mathcal{L} := -\partial_x (c^2 - 2\eta)\partial_x + (2\eta'' - 1).}$$

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The conserved energy quadratic form is

$$\langle \mathcal{L}\zeta, \zeta\rangle = \oint \left[(c^2 - 2\eta)(\partial_x\zeta)^2 + (2\eta'' - 1)\zeta^2\right] dx.$$

- The spectrum $\sigma(\mathcal{L})$ consists of isolated eigenvalues.
- We have $0 \in \sigma(\mathcal{L})$ because $\mathcal{L}\eta' = 0$.
- 0 is the third eigenvalue in the spectrum $\sigma(\mathcal{L}) = \{\lambda_1, \lambda_2, 0, \lambda_4, \dots\}$ (shown by the period function).

$\langle \mathcal{L}\zeta, \zeta\rangle$ is coercive under constraints $\langle 1, \zeta\rangle = 0$ and $\langle \eta', \zeta'\rangle = 0$, which are invariant due to conservation of M and Q.

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Theorem (S. Locke–D. P., JFM, 2025)

For every initial data $\zeta_0 \in H^1_{\mathrm{per}}(\mathbb{T})$ satisfying the two constraints, there exists a unique solution $\zeta \in C^0(\mathbb{R}, H^1_{\mathrm{per}}(\mathbb{T}))$ and a unique $a \in C^0(\mathbb{R}, \mathbb{R})$ such that

$$\|\zeta(\cdot, t) - a(t)\eta'\|_{H^1_{\mathrm{per}}} \leq C\|\zeta_0\|_{H^1_{\mathrm{per}}}, \quad |a'(t)| \leq C\|\zeta_0\|_{H^1_{\mathrm{per}}}, \quad t \in \mathbb{R},$$

where $C > 0$ is independent of ζ_0.

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The linear evolution $2c\partial_t\zeta = -J\mathcal{L}\zeta$ is defined by

$$\boxed{J = \Pi_0 \partial_x^{-1} \Pi_0, \quad \mathcal{L} := -\partial_x (c^2 - 2\eta)\partial_x + (2\eta'' - 1).}$$

- Linear stability does not imply nonlinear stability because we have no local well-posedness in $H^1_{\rm per}(\mathbb{T})$ but the $W^{1,\infty}$-norm of the perturbation ζ is not controlled in the time evolution.
- For nonlinear stability in the CH equation, one needs to use the additional variable $m := \zeta - \zeta_{xx}$ to control the solution either in $H^1_{\rm per}(\mathbb{T}) \cap W^{1,\infty}_{\rm per}(\mathbb{T})$ or in $H^3_{\rm per}(\mathbb{T})$ as in [S. Lafortune, D.P, Physica D, 2022]

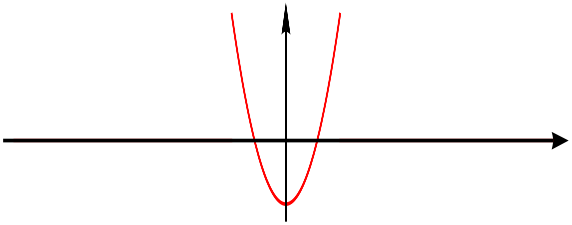
4. Instability of peaked waves

We have the evolution equation

$$\boxed{2c\partial_t \eta = (c^2 - 2\eta)\partial_x \eta + \Pi_0 \partial_x^{-1} \Pi_0 \left[(\partial_x \eta)^2 + \eta \right]}$$

for the peaked traveling wave with the unique speed $c = c_* = \frac{\pi}{2\sqrt{2}}$ and the peaked profile

$$\eta_*(x) = \frac{1}{16}(\pi^2 - 4\pi|x| + 2x^2), \qquad x \in [-\pi, \pi].$$



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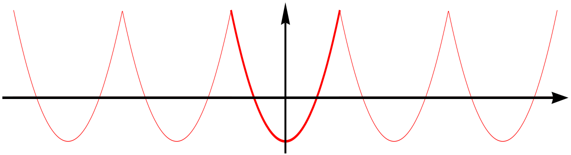
for the peaked traveling wave with the unique speed $c = c_* = \frac{\pi}{2\sqrt{2}}$ and the peaked profile

$$\eta_*(x) = \frac{1}{16}(\pi^2 - 4\pi|x| + 2x^2), \qquad x \in [-\pi, \pi].$$

which is periodically continued on $\mathbb{T}$.



Proper decomposition for linearization

The evolution equation

$$\boxed{2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta + \Pi_0\partial_x^{-1}\Pi_0\left[(\partial_x\eta)^2 + \eta\right]}$$

is close to the inviscid Burgers (Hopf) equation

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta.$$

Proper decomposition for linearization

The evolution equation

$$\boxed{2c\partial_t \eta = (c^2 - 2\eta)\partial_x \eta + \Pi_0 \partial_x^{-1} \Pi_0 \left[(\partial_x \eta)^2 + \eta \right]}$$

is close to the inviscid Burgers (Hopf) equation

$$2c\partial_t \eta = (c^2 - 2\eta)\partial_x \eta.$$

If $\eta \in C^0((-\tau_0, \tau_0), H^1_{\mathrm{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}))$ is a local solution and there exists $\xi(t)$ such that

$$\lim_{x \to \xi(t)^-} \partial_x \eta(t,x) \neq \lim_{x \to \xi(t)^+} \partial_x \eta(t,x), \quad t \in (-\tau_0, \tau_0),$$

then $\xi \in C^1((-\tau_0, \tau_0))$ and

$$2c \frac{d\xi}{dt} = -(c^2 - 2\eta(t, \xi(t))), \quad t \in (-\tau_0, \tau_0).$$

Proper decomposition for linearization

The evolution equation

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$$2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta.$$

Assuming that $\eta \in C^0((-\tau_0, \tau_0), H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}))$ has a single peak at $x = \xi(t)$, we consider the perturbation $\zeta(t, x)$ as

$$\eta(t, x) = \eta_*(x - \xi(t)) + \zeta(t, x - \xi(t)).$$

This gives the evolution equation

$$\begin{aligned} 2c_*\partial_t\zeta = {} & (c_*^2 - 2\eta_*)\partial_x\zeta - 2(\zeta - \zeta|_{x=0})(\eta'_* + \partial_x\zeta) \\ & + \Pi_0\partial_x^{-1}\Pi_0\left[\zeta + 2\eta'_*\partial_x\zeta + (\partial_x\zeta)^2\right], \end{aligned}$$

Proper decomposition for linearization

The evolution equation

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After integration by parts $\zeta + 2\eta_*'\partial_x\zeta = 2\partial_x(\eta_*'\zeta) + \frac{1}{2}\zeta + \pi\delta_0\zeta$, and truncation, the linearized evolution equation takes the better form:

$$2c_*\partial_t\zeta = (c_*^2 - 2\eta_*)\partial_x\zeta - \frac{1}{\pi}\oint\eta_*'\zeta dx + \frac{1}{2}\Pi_0\partial_x^{-1}\Pi_0\zeta,$$

where both $\oint\zeta dx$ and $\zeta|_{x=0}$ are constant in t and satisfy the constraint $\zeta|_{x=0} = -\frac{1}{2\pi}\oint\zeta dx$.

Proper linearized operator

The linearized evolution equation

$$\boxed{2c_*\partial_t\zeta = (c_*^2 - 2\eta_*)\partial_x\zeta - \frac{1}{\pi}\oint \eta_*'\zeta dx + \frac{1}{2}\Pi_0\partial_x^{-1}\Pi_0\zeta}$$

is defined by the operator $A : \mathrm{Dom}(A) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$ s.t.

$$Af := (c_*^2 - 2\eta_*)\partial_x f - \frac{1}{\pi}\oint \eta_*' f dx + \frac{1}{2}\Pi_0\partial_x^{-1}\Pi_0 f,$$

where $\mathrm{Dom}(A) := \left\{ f \in L^2(\mathbb{T}) : \ \ (c_*^2 - 2\eta_*)f' \in L^2(\mathbb{T}) \right\}$.

Proper linearized operator

The linearized evolution equation

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where $\mathrm{Dom}(A) := \left\{f \in L^2(\mathbb{T}) : \ (c_*^2 - 2\eta_*)f' \in L^2(\mathbb{T})\right\}$.

For local well-posedness, we should consider
$A : H^1_{\mathrm{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}) \subset L^2(\mathbb{T}) \cap L^\infty(\mathbb{T}) \to L^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$,
where $H^1_{\mathrm{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ is embedded into $\mathrm{Dom}(A)$.

Spectral instability result

Theorem (F. Natali, D.P., S. Wang, JNLW 1 (2025) e10)

The spectrum of $A : \mathrm{Dom}(A) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$ *completely covers the closed vertical strip given by*

$$\sigma(A) = \left\{ \lambda \in \mathbb{C} : \quad -\frac{\pi}{4} \leq \mathrm{Re}(\lambda) \leq \frac{\pi}{4} \right\}.$$

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In fact, we show

$$\sigma_p(A) = \left\{ \lambda \in \mathbb{C} : \quad -\frac{\pi}{4} < \mathrm{Re}(\lambda) < \frac{\pi}{4} \right\},$$

$$\rho(A) = \left\{ \lambda \in \mathbb{C} : \quad |\mathrm{Re}(\lambda)| > \frac{\pi}{4} \right\},$$

so that $\sigma_c(A) = \sigma(A) \backslash \sigma_p(A)$.

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To find $\sigma_p(A)$, we analyze $Af = \lambda f$, $f \in \mathrm{Dom}(A)$:

$$\frac{1}{4} x(2\pi - x) f'(x) + \frac{1}{4\pi} \int_0^{2\pi} (\pi - x) f(x) dx + \frac{1}{2} \Pi_0 \partial_x^{-1} \Pi_0 f = \lambda f(x),$$

with the constraint $\lambda \int_0^{2\pi} f(x) dx = 0$.

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Since $f \in C^\infty(0, 2\pi)$, the spectral problem is the ODE:

$$\frac{1}{4}x(2\pi - x)f''(x) + \frac{1}{2}(\pi - x)f'(x) + \frac{1}{2}f(x) - \frac{1}{4\pi}\underbrace{\int_0^{2\pi} f(x)dx}_{=0} = \lambda f'(x),$$

with two solutions $f_1(x) = 2\lambda - \pi + x$ and $f_2(x) \sim x^{\frac{2\lambda}{\pi}}$, $x \to 0^+$.

Spectral instability result

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The spectrum of $A : \mathrm{Dom}(A) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$ *completely covers the closed vertical strip given by*

$$\sigma(A) = \left\{ \lambda \in \mathbb{C} : \quad -\frac{\pi}{4} \leq \mathrm{Re}(\lambda) \leq \frac{\pi}{4} \right\}.$$

For

$$\sigma_p(A) = \left\{ \lambda \in \mathbb{C} : \quad -\frac{\pi}{4} < \mathrm{Re}(\lambda) < \frac{\pi}{4} \right\},$$

both $f_1, f_2 \in \mathrm{Dom}(A)$ and $c_1 f_1(x) + c_2 f_2(x)$ satisfies the constraint $\int_0^{2\pi} f(x) dx = 0$.

Nonlinear instability result

Theorem (F. Natali, D.P., S. Wang, JNLW 1 (2025) e10)

For every $\delta > 0$ *there exists* $\zeta_0 \in H^1_{\rm per}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ *satisfying*

$$\|\zeta_0\|_{H^1_{\rm per}} \leq \delta^2, \quad \|\zeta_0\|_{W^{1,\infty}} \leq \delta,$$

such that the unique local solution ζ *satisfies* $\|\zeta(t_0,\cdot)\|_{W^{1,\infty}} = 1$*.*

Nonlinear instability result

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such that the unique local solution ζ satisfies $\|\zeta(t_0,\cdot)\|_{W^{1,\infty}} = 1$.

The nonlinear evolution equation is

$$2c_*\partial_t\zeta = (c_*^2-2\eta_*)\partial_x\zeta-2(\zeta-\zeta|_{x=0})\partial_x\zeta-\frac{1}{\pi}\langle\eta_*',\zeta\rangle+\frac{1}{2}\Pi_0\partial_x^{-1}\Pi_0\left[\zeta+2(\partial_x\zeta)^2\right].$$

with two conserved quantities

$$\oint \zeta dx \quad \text{and} \quad \zeta|_{x=0}+\frac{1}{\pi}\oint(\partial_x\zeta)^2dx = C_0.$$

Nonlinear instability result

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such that the unique local solution ζ *satisfies* $\|\zeta(t_0,\cdot)\|_{W^{1,\infty}} = 1$.

Characteristic curves for $x = X(t,s)$:

$$\left\{ \begin{array}{l} 2c_* \partial_t X(t,s) = -(c_*^2 - 2\eta_*(X)) + 2(\zeta(t,X) - \zeta(t,0)), \\ X(0,s) = s. \end{array} \right.$$

and evolution of $Z(t,s) := \zeta(t, X(t,s))$ along the curves

$$\left\{ \begin{array}{l} 2c_* \partial_t Z(t,s) = -\frac{1}{\pi} \langle \eta_*', \zeta \rangle + \frac{1}{2} \Pi_0 \partial_x^{-1} \Pi_0 (\zeta + 2(\partial_x \zeta)^2), \\ Z(0,s) = \zeta_0(s), \end{array} \right.$$

Nonlinear instability result

Theorem (F. Natali, D.P., S. Wang, JNLW 1 (2025) e10)

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$$\|\zeta_0\|_{H^1_{\rm per}} \leq \delta^2, \quad \|\zeta_0\|_{W^{1,\infty}} \leq \delta,$$

such that the unique local solution ζ satisfies $\|\zeta(t_0,\cdot)\|_{W^{1,\infty}} = 1$.

Assuming $\zeta_0 \in C^1(0,2\pi)$, we get for $V(t,s) := \partial_x \zeta(t, X(t,s))$:

$$\left\{ \begin{array}{l} 2c_* \partial_t V(t,s) = -2\eta_*'(X) V - V^2 + \frac{1}{2}(Z(t,s) + Z(t,0)), \\ V(0,s) = \zeta_0'(s), \end{array} \right.$$

with the one-sided limit to the peak at $V_0(t) := \lim\limits_{s \to 0^+} V(t,s)$:

$$2c_* V_0'(t) = \frac{\pi}{2} V_0(t) - V_0^2(t) + Z(t,0) \leq \frac{\pi}{2} V_0(t) + C_0.$$

Nonlinear instability result

Theorem (F. Natali, D.P., S. Wang, JNLW 1 (2025) e10)

For every $\delta > 0$ *there exists* $\zeta_0 \in H^1_{\mathrm{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ *satisfying*

$$\|\zeta_0\|_{H^1_{\mathrm{per}}} \le \delta^2, \quad \|\zeta_0\|_{W^{1,\infty}} \le \delta,$$

such that the unique local solution ζ *satisfies* $\|\zeta(t_0,\cdot)\|_{W^{1,\infty}} = 1$.

This gives the instability in the $W^{1,\infty}$-norm:

$$V_0(t) \le \left(V_0(0) + \frac{2}{\pi} C_0\right) e^{\frac{\pi t}{4c_*}}.$$

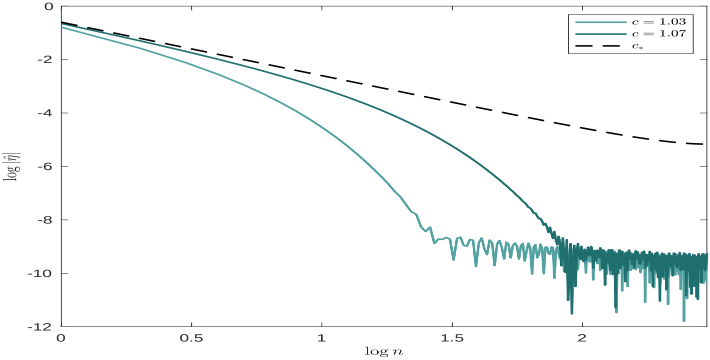
for $-\delta < V_0(0) < -\frac{2}{\pi}|C_0|$, where $|C_0| \lesssim \delta^2$.

5. Convergence of smooth waves to the peaked wave.

The wave profile is found from the first-order quadrature:

$$\left\{ \begin{array}{l} \left(\dfrac{d\eta}{dx}\right)^2 = \dfrac{2\mathcal{E} - \eta^2}{c^2 - 2\eta}, \\ \eta(\pm\pi) = -\sqrt{2\mathcal{E}}. \end{array} \right.$$

Coefficients of Fourier series decay exponentially for smooth waves and algebraically for the peaked wave.

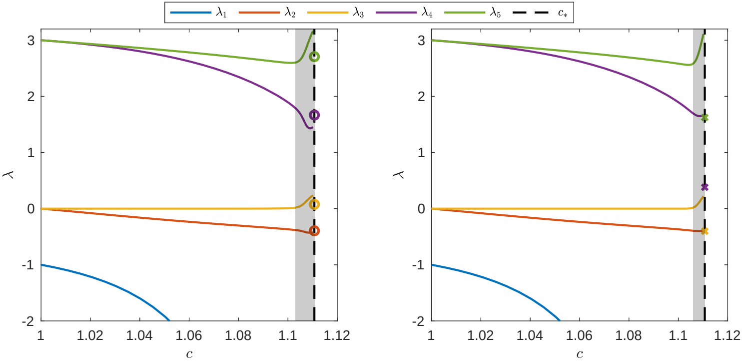


Eigenvalues of the self-adjoint operator.

The spectrum of $\mathcal{L} : H^2_{\text{per}}(\mathbb{T}) \to L^2(\mathbb{T})$ is purely discrete:

$$\mathcal{L} = -\partial_x (c^2 - 2\eta)\partial_x + (2\eta'' - 1).$$

The lowest eigenvalue diverges in the limit of peaked waves and the numerical accuracy becomes poor.

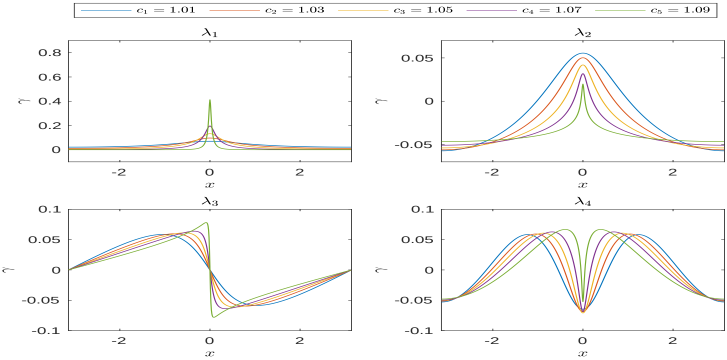


Eigenfunctions of the self-adjoint operator.

The spectrum of $\mathcal{L} : H^2_{\text{per}}(\mathbb{T}) \to L^2(\mathbb{T})$ is purely discrete:

$$\mathcal{L} = -\partial_x (c^2 - 2\eta)\partial_x + (2\eta'' - 1).$$

Eigenfunctions become peaked in the limit of peaked waves.

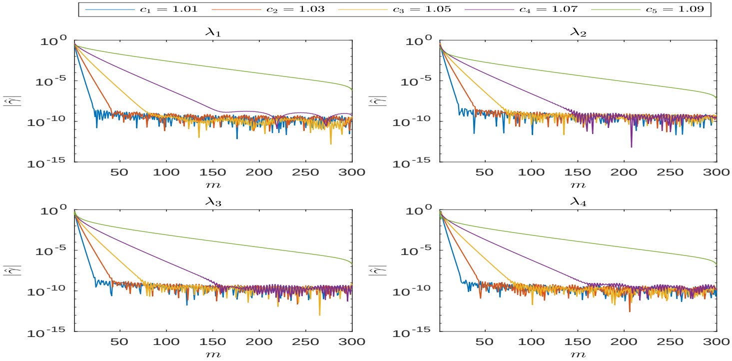


Eigenfunctions in the Fouier space

The spectrum of $\mathcal{L} : H^2_{\mathrm{per}}(\mathbb{T}) \to L^2(\mathbb{T})$ is purely discrete:

$$\mathcal{L} = -\partial_x(c^2 - 2\eta)\partial_x + (2\eta'' - 1).$$

Their Fourier spectrum decays slowly for nearly-peaked waves.

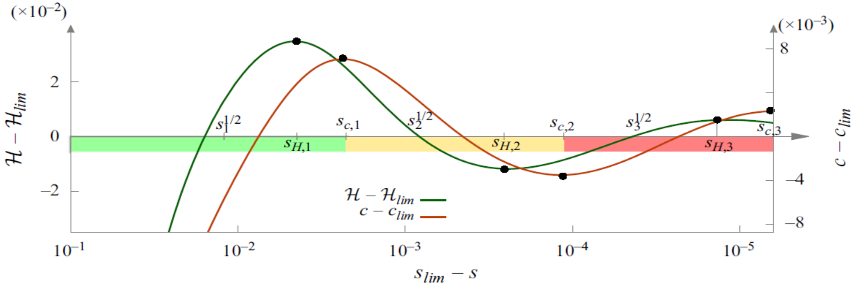


No convergence along the family of smooth waves?

Recall the Babenko equation for deep fluid

$$(c^2|\partial_x| - 1)\eta = \frac{1}{2}|\partial_x|\eta^2 + \eta|\partial_x|\eta,$$

with ∞-many oscillations for c in $(1, c_{\max})$ with $c_* \approx 1.0922$.



6. Summary

We considered the following model for $\eta = \eta(t,x)$:

$$\boxed{2c\partial_x\partial_t\eta = (c^2 - 2\eta)\partial_x^2\eta - (\partial_x\eta)^2 + \eta}$$

with x defined in the 2π-periodic domain $\mathbb{T}$ and $c > 0$ being a parameter for the wave speed.

- The smooth waves are linearly stable in the time evolution.
- The peaked wave is linearly and nonlinearly unstable.
- The cusped waves belong to $H^1_{\rm per}$ but do not belong to $H^1_{\rm per} \cap W^{1,\infty}$, where local well-posedness is shown. Their stability analysis is open.