Instability of the peaked traveling wave in a local model for shallow water waves

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Outline of the talk

1. The model and motivations.
2. Conserved quantities and local well-posedness.
3. Smooth and peaked traveling waves.
4. Instability of the peaked traveling wave.
5. Convergence to the peaked wave along the family of smooth waves.

The talk is based on the joint works with

- Spencer Locke (PhD at University of Michigan, USA),
- Shuoyang Wang (BSc at McMaster University, Canada),
- Fabio Natali (University of Maringa, Brazil).

1. The model and motivations.

We consider the following PDE for $\eta = \eta(t, x)$:

$$\boxed{2c\partial_x \partial_t \eta = (c^2 - 2\eta)\partial_x^2 \eta - (\partial_x \eta)^2 + \eta}$$

with x defined in the 2π-periodic domain $\mathbb{T}$ and $c > 0$ being a parameter for the wave speed.

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- High-frequency limit of the Camassa–Holm equation

$$u_T - u_{TXX} + ku_X + 3uu_X = 2u_Xu_{XX} + uu_{XXX},$$

for solutions of the form $u(T,X) = 2\eta(t,x)$ with $t = 2c\varepsilon^{-1}T$, $x = \varepsilon^{-1}(X - c^2T)$, $k = \varepsilon^{-2}$ and $\varepsilon \to 0$.

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- Extension of the Hunter–Saxton equation

$$(u_T + uu_X)_X = \frac{1}{2}(u_X)^2$$

for solutions of the form $u(T,X) = 2\eta(t,x)$ with $t = 2cT$, $x = X + c^2T$, without the last term.

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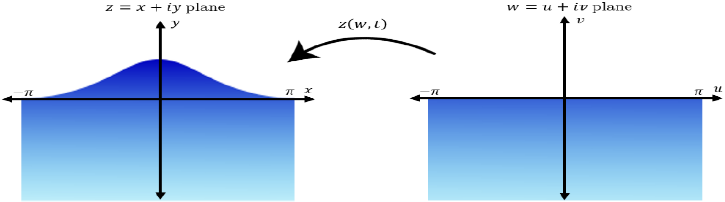
with x defined in the 2π-periodic domain $\mathbb{T}$ and $c > 0$ being a parameter for the wave speed.

- Truncation of the full water wave equation in conformal variables with η being the surface elevation.



S.Locke–D.P., JFM 2025

Recent numerical results

Existence of smooth Stokes waves of large amplitudes was obtained numerically with a high precision.

S. Dyachenko, P. Lushnikov, A. Korotkevich (2016)

S. Dyachenko, V. Hur, D. Silantyev (2023)

Spectral stability of smooth Stokes waves of large amplitudes was explored numerically both for co-periodic and localized perturbations.

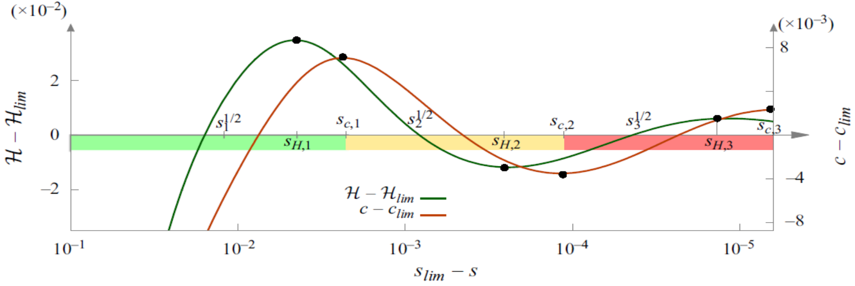
S. Dyachenko, A. Semenova (2023)

A. Korotkevich, P. Lushnikov, A. Semenova, S. Dyachenko (2023)

B. Deconinck, S. Dyachenko, A. Semenova (2024)

Recent numerical results

The wave energy H and the wave speed c oscillate as functions of the wave steepness s towards the limiting wave with the peaked profile:

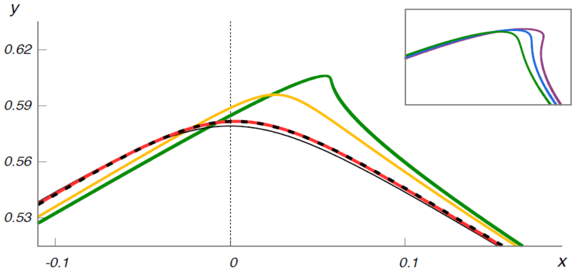


Conjectures based on the numerical results:

- $\exists\ \infty$-many oscillations of wave energy and speed.
- The family of smooth Stokes waves converge to the limiting wave of the peaked profile.

Recent numerical results

The wave profile becomes steep and breaks due to instability:



The traveling wave of the peaked profile is believed to exist for a unique value of $c = c_*$. However, the proof of uniqueness of c_* and the convergence as $c \to c_*$ is open. For model equations (fractional KdV, Whitham), the proof of uniqueness of c_* is in

J. Dahne, J. Diff. Eqs. 401 (2024) 550

M. Ehrnström, O.I.H. Mæhlen, K. Varholm, Ann. Inst. H. Poincaré C (2025)

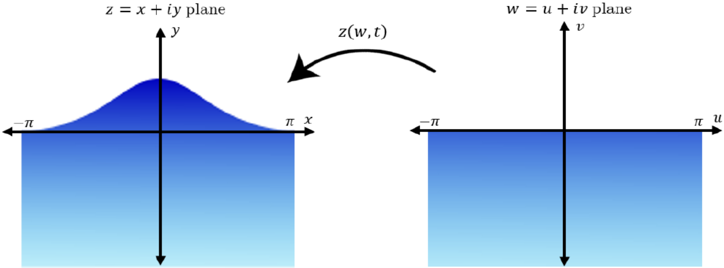
Babenko's equation for traveling (Stokes) waves

Traveling waves $\eta(u,t) = \eta(u - ct)$ with the zero-mean profile η satisfy a scalar pseudo–differential equation

$$\boxed{(c^2\mathcal{K}_h - 1)\eta = \frac{1}{2}\mathcal{K}_h\eta^2 + \eta\mathcal{K}_h\eta,}$$

where the self-adjoint operator $\mathcal{K}_h$ is defined by

$$\widehat{(\mathcal{K}_h)}_n = \begin{cases} n\coth(hn), & n \in \mathbb{Z}\backslash\{0\}, \\ 0, & n = 0. \end{cases}$$



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Existence of traveling waves with both smooth and peaked profiles is defined by solutions of Babenko's equation.

K. Babenko, Russian Academy of Sciences 294 (1987) 1033

⋮

S. Locke & D.P., Appl. Math. Lett. 161 (2025) 109359

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The zero-mean constraint on η in the physical coordinate becomes the quadratic constraint in the conformal coordinate:

$$\oint \eta + \oint \eta\mathcal{K}_h\eta = 0.$$

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In the deep-water limit $h \to \infty$, we have $\mathcal{K}_h \to |\partial_x|$ and Babenko's equation becomes the "stationary BO" equation

$$(c^2 |\partial_x| - 1)\eta = \frac{1}{2}|\partial_x| \eta^2 + \eta |\partial_x| \eta,$$

with ∞-many oscillations for c in $(1, c_{\max})$ with $c_* \approx 1.0922$.

Babenko's equation for traveling (Stokes) waves

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$$\widehat{(\mathcal{K}_h)}_n = \begin{cases} n \coth(hn), & n \in \mathbb{Z}\backslash\{0\}, \\ 0, & n = 0. \end{cases}$$

In the shallow-water limit $h \to 0$, we replace $\mathcal{K}_h$ by $-\partial_x^2$ and Babenko's equation becomes the stationary equation

$$(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0,$$

which is equivalent to the steady model considered here.

2. Conserved quantities and local well-posedness.

Assume a smooth solution $\eta \in C^0((-\tau_0, \tau_0), H^s_{\mathrm{per}}(\mathbb{T}))$, $s > \frac{3}{2}$:

$$\boxed{2c\partial_x\partial_t\eta = (c^2 - 2\eta)\partial_x^2\eta - (\partial_x\eta)^2 + \eta}$$

The model has the three (basic) conserved quantities:

- Mass

$$M(\eta) = \oint \eta dx$$

- Momentum

$$Q(\eta) = \frac{1}{2}\oint (\partial_x\eta)^2 dx$$

- Energy

$$H(\eta) = \frac{1}{2}\oint \left[\eta^2 + 2\eta(\partial_x\eta)^2\right] dx.$$

Moreover, it admits a nontrivial constraint

$$M(\eta) + 2Q(\eta) = \oint \left[\eta + (\partial_x\eta)^2\right] dx = 0.$$

Local well-posedness of the initial-value problem

Integrating once in x,

$$2c\partial_x\partial_t\eta = (c^2 - 2\eta)\partial_x^2\eta - (\partial_x\eta)^2 + \eta$$

we can write the evolution problem

$$\boxed{2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta + \Pi_0\partial_x^{-1}\Pi_0\left[(\partial_x\eta)^2 + \eta\right]}$$

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The mapping

$$\Pi_0\partial_x^{-1}\Pi_0\left[(\partial_x\eta)^2 + \eta\right] : H^1_{\text{per}} \cap W^{1,\infty} \to H^1_{\text{per}} \cap W^{1,\infty}$$

is bounded on every bounded subset.

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The inviscid Burgers equation

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta$$

is locally well-posed in $H^1_{\text{per}} \cap W^{1,\infty}$
(e.g., the method of characteristics).

Local well-posedness of the initial-value problem

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- The initial-value problem is locally well-posed in $H^1_{\text{per}} \cap W^{1,\infty}$.

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where $\Pi_0 : L^2(\mathbb{T}) \to L^2(\mathbb{T})|_{\{1\}^\perp}$.

- The initial-value problem for the evolution probem is also locally well-posed in H^s_{per}, $s > \frac{3}{2}$, which is embedded into $H^1_{\text{per}} \cap W^{1,\infty}$.

3. Smooth and peaked traveling waves.

Traveling waves are defined by solutions of the 2^{nd}-order ODE:

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Theorem (S. Locke–D. P., JFM, 2025)

There exist $c_* := \frac{\pi}{2\sqrt{2}}$ and $c_\infty \in (c_*, \infty)$ such that the ODE admits a unique solution with the profile $\eta \in C^\infty_{\text{per}}(\mathbb{T})$ for every $c \in (1, c_*)$ s.t.

$$\|\eta\|_{L^\infty} \to 0 \quad \text{as} \quad c \to 1$$

and a solution with the profile $\eta \in C^0_{\text{per}}(\mathbb{T})$ for every $c \in (c_*, c_\infty)$ satisfying for some $A(c) > 0$,

$$\eta(x) = \frac{c^2}{2} - A(c)|x|^{2/3} + \mathcal{O}(|x|) \quad \text{as} \quad x \to 0.$$

3. Smooth and peaked traveling waves.

Traveling waves are defined by solutions of the $2^{\rm nd}$-order ODE:

$$(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0, \qquad x \in \mathbb{T}.$$

- The two continuous families meet at $c = c_* = \frac{\pi}{2\sqrt{2}}$, where the profile $\eta \in C^0_{\rm per}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ is peaked:

$$\eta(x) = \frac{1}{16}(\pi^2 - 4\pi|x| + 2x^2), \qquad x \in [-\pi, \pi].$$

with

$$\|\eta\|_{L^\infty} = \eta(0) = \frac{\pi^2}{16} = \frac{c^2}{2}.$$

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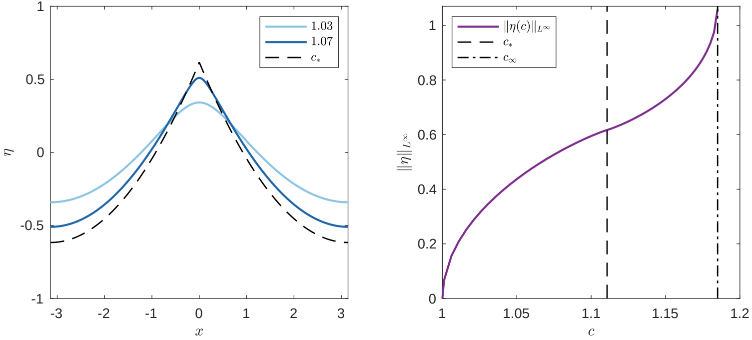

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$$\boxed{(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0, \qquad x \in \mathbb{T}.}$$

- The highest amplitude

$$\max_{x \in \mathbb{T}} \eta(x) = \eta(0) = \frac{c^2}{2}$$

  follows from Bernoulli's principle of hydrodynamics.

- The $|x|^{2/3}$ singularity in the conformal coordinate corresponds to Stokes' law of the 120^0 angle in the physical coordinate.

Existence

Smooth solutions of $\boxed{(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0}$ are level curves of

$$E(\eta, \eta') := \frac{1}{2}(c^2 - 2\eta)(\eta')^2 + \frac{1}{2}\eta^2 = \mathcal{E}$$

on the phase plane (η, η').

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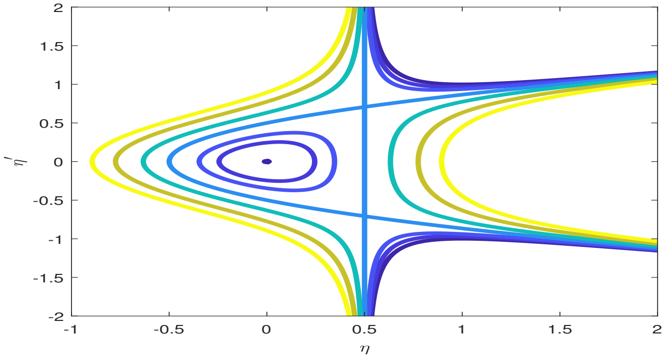

Linear stability

The evolution equation

$$\boxed{2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta + \Pi_0\partial_x^{-1}\Pi_0\left[(\partial_x\eta)^2 + \eta\right]}$$

has the Hamiltonian formulation

$$2c\partial_t\eta = J\nabla\left[H(\eta) - c^2 Q(\eta)\right], \quad J := \Pi_0\partial_x^{-1}\Pi_0.$$

Here H is the energy and Q is the momentum given by

$$Q(\eta) = \frac{1}{2}\oint(\partial_x\eta)^2 dx, \quad H(\eta) = \frac{1}{2}\oint\left[\eta^2 + 2\eta(\partial_x\eta)^2\right] dx.$$

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The profile $\eta \in C^\infty_{\text{per}}(\mathbb{T})$ for smooth traveling waves is a critical point of $H - c^2 Q$ so that the perturbation $\zeta \in H^1_{\text{per}} \cap W^{1,\infty}$ satisfies the linearized equation

$$2c\partial_t\zeta = -J\mathcal{L}\zeta, \quad \mathcal{L} := -\partial_x(c^2 - 2\eta)\partial_x + (2\eta'' - 1).$$

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$$2c\partial_t\zeta = -J\mathcal{L}\zeta, \quad \mathcal{L} := -\partial_x(c^2 - 2\eta)\partial_x + (2\eta'' - 1).$$

The spectrum of $\mathcal{L} : H^2_{\rm per}(\mathbb{T}) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is purely discrete. The coefficients of $\mathcal{L}$ are singular in the limit of peaked wave since $2\eta''(x) - 1 \to -\frac{1}{2} - \pi\delta_0$ with Dirac δ_0.

Linear stability of smooth waves

The linear evolution $2c\partial_t \zeta = -J\mathcal{L}\zeta$ is defined by

$$\boxed{J = \Pi_0 \partial_x^{-1} \Pi_0, \quad \mathcal{L} := -\partial_x (c^2 - 2\eta)\partial_x + (2\eta'' - 1).}$$

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- The spectral problem

$$\Pi_0\partial_x^{-1}\Pi_0\mathcal{L}\zeta = \lambda\zeta, \quad \zeta \in H^2_{\rm per}(\mathbb{T})$$

coincides with the spectral problem

$$\mathcal{L}\zeta = \lambda\partial_x\zeta, \quad \zeta \in H^2_{\rm per}(\mathbb{T})$$

studied in [M. Stanislavova–A. Stefanov, CMP, 2016] .

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The linear evolution $2c\partial_t \zeta = -J\mathcal{L}\zeta$ is defined by

$$J = \Pi_0 \partial_x^{-1} \Pi_0, \quad \mathcal{L} := -\partial_x (c^2 - 2\eta) \partial_x + (2\eta'' - 1).$$

The conserved energy quadratic form

$$\langle \mathcal{L}\zeta, \zeta \rangle = \oint \left[(c^2 - 2\eta)(\partial_x \zeta)^2 + (2\eta'' - 1)\zeta^2 \right] dx,$$

is defined by the self-adjoint operator $\mathcal{L} : H^2_{\rm per}(\mathbb{T}) \to L^2(\mathbb{T})$:

- The spectrum $\sigma(\mathcal{L})$ consists of isolated eigenvalues.
- We have $0 \in \sigma(\mathcal{L})$ because $\mathcal{L}\eta' = 0$.
- 0 is the third eigenvalue in the spectrum $\sigma(\mathcal{L}) = \{\lambda_1, \lambda_2, 0, \lambda_4, \dots\}$ (shown by the period function).

$\langle \mathcal{L}\zeta, \zeta \rangle$ is coercive under constraints $\langle 1, \zeta \rangle = 0$ and $\langle \eta', \zeta' \rangle = 0$.

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The linear evolution $2c\partial_t\zeta = -J\mathcal{L}\zeta$ is defined by

$$\boxed{J = \Pi_0\partial_x^{-1}\Pi_0, \quad \mathcal{L} := -\partial_x(c^2 - 2\eta)\partial_x + (2\eta'' - 1).}$$

Two constraints follow from the conservation of mass M and momentum Q: $\langle 1, \zeta\rangle = 0$ and $\langle \eta', \zeta'\rangle = 0$.

Theorem (S. Locke–D. P., JFM, 2025)

For every initial data $\zeta_0 \in H^1_{\rm per}(\mathbb{T})$ satisfying the two constraints, there exists a unique solution $\zeta \in C^0(\mathbb{R}, H^1_{\rm per}(\mathbb{T}))$ and a unique $a \in C^0(\mathbb{R}, \mathbb{R})$ such that

$$\|\zeta(\cdot,t) - a(t)\eta'\|_{H^1_{\rm per}} \leq C\|\zeta_0\|_{H^1_{\rm per}}, \quad |a'(t)| \leq C\|\zeta_0\|_{H^1_{\rm per}}, \quad t \in \mathbb{R},$$

where $C > 0$ is independent of ζ_0.

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- Linear stability does not imply nonlinear stability because we have no local well-posedness in $H^1_{\text{per}}(\mathbb{T})$ but the $W^{1,\infty}$-norm of the perturbation ζ is not controlled in the time evolution.
- For nonlinear stability in the CH equation, one needs to use the additional variable $m := \zeta - \zeta_{xx}$ to control the solution either in $H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}_{\text{per}}(\mathbb{T})$ or in $H^3_{\text{per}}(\mathbb{T})$ as in [S. Lafortune, D.P, Physica D, 2022]

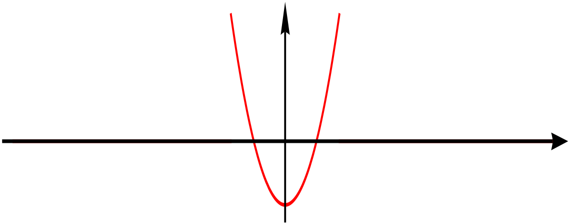
4. Instability of peaked waves

We have the evolution equation

$$\boxed{2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta + \Pi_0\partial_x^{-1}\Pi_0\left[(\partial_x\eta)^2 + \eta\right]}$$

but TW has a peaked profile $\eta_* \in C^0_{\rm per}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ for $c = c_* := \frac{\pi}{2\sqrt{2}}$,

$$\eta_*(x) = \frac{1}{16}(\pi^2 - 4\pi|x| + 2x^2), \qquad x \in [-\pi, \pi].$$



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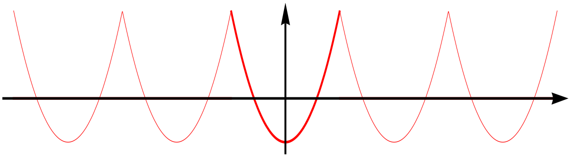
which is periodically continued on $\mathbb{T}$.



Uniqueness of the peaked periodic wave for $c = c_*$ was proven in [A. Geyer & D.P, SIMA, 2019] [G. Bruell & Dhara, Indiana Math. J. 2021]

Proper decomposition for linearization

The evolution equation

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is close to the inviscid Burgers (Hopf) equation

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is close to the inviscid Burgers (Hopf) equation

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta.$$

If $\eta \in C^0((-\tau_0, \tau_0), H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}))$ is a local solution and there exists $\xi(t)$ such that

$$\lim_{x\to\xi(t)^-} \partial_x\eta(t,x) \neq \lim_{x\to\xi(t)^+} \partial_x\eta(t,x), \quad t \in (-\tau_0, \tau_0),$$

then $\xi \in C^1((-\tau_0, \tau_0))$ and

$$2c\frac{d\xi}{dt} = -(c^2 - 2\eta(t,\xi(t))), \quad t \in (-\tau_0, \tau_0).$$

Proper decomposition for linearization

The evolution equation

$$\boxed{2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta + \Pi_0\partial_x^{-1}\Pi_0\left[(\partial_x\eta)^2 + \eta\right]}$$

is close to the inviscid Burgers (Hopf) equation

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta.$$

Assuming that $\eta \in C^0((-\tau_0,\tau_0), H^1_{\rm per}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}))$ has a single peak at $x = \xi(t)$, we consider the perturbation $\zeta(t,x)$ as

$$\eta(t,x) = \eta_*(x - \xi(t)) + \zeta(t, x - \xi(t)).$$

This gives the evolution equation

$$\begin{aligned} 2c_*\partial_t\zeta = & \, (c_*^2 - 2\eta_*)\partial_x\zeta - 2(\zeta - \zeta|_{x=0})(\eta_*' + \partial_x\zeta) \\ & + \Pi_0\partial_x^{-1}\Pi_0\left[\zeta + 2\eta_*'\partial_x\zeta + (\partial_x\zeta)^2\right], \end{aligned}$$

Proper decomposition for linearization

The evolution equation

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$$2c\partial_t\eta = (c^2 - 2\eta)\partial_x\eta.$$

After integration by parts $\zeta + 2\eta_*'\partial_x\zeta = 2\partial_x(\eta_*'\zeta) + \frac{1}{2}\zeta + \pi\delta_0\zeta$, and truncation, the linearized evolution equation takes the better form:

$$2c_*\partial_t\zeta = (c_*^2 - 2\eta_*)\partial_x\zeta - \frac{1}{\pi}\oint \eta_*'\zeta dx + \frac{1}{2}\Pi_0\partial_x^{-1}\Pi_0\zeta,$$

where both $\oint \zeta dx$ and $\zeta|_{x=0}$ are constant in t and satisfy the constraint $\zeta|_{x=0} = -\frac{1}{2\pi}\oint \zeta dx$.

Proper linearized operator

The linearized evolution equation

$$\boxed{2c_*\partial_t\zeta = (c_*^2 - 2\eta_*)\partial_x\zeta - \frac{1}{\pi}\oint \eta_*'\zeta dx + \frac{1}{2}\Pi_0\partial_x^{-1}\Pi_0\zeta}$$

is defined by the operator $A : \mathrm{Dom}(A) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$ s.t.

$$Af := (c_*^2 - 2\eta_*)\partial_x f - \frac{1}{\pi}\oint \eta_*' f dx + \frac{1}{2}\Pi_0\partial_x^{-1}\Pi_0 f,$$

where $\mathrm{Dom}(A) := \left\{f \in L^2(\mathbb{T}) : \ (c_*^2 - 2\eta_*)f' \in L^2(\mathbb{T})\right\}$.

Proper linearized operator

The linearized evolution equation

$$\boxed{2c_*\partial_t\zeta = (c_*^2 - 2\eta_*)\partial_x\zeta - \frac{1}{\pi}\oint \eta_*'\zeta dx + \frac{1}{2}\Pi_0\partial_x^{-1}\Pi_0\zeta}$$

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where $\mathrm{Dom}(A) := \left\{f \in L^2(\mathbb{T}) : \ (c_*^2 - 2\eta_*)f' \in L^2(\mathbb{T})\right\}$.

For local well-posedness, we should consider
$A : H^1_{\mathrm{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}) \subset L^2(\mathbb{T}) \cap L^\infty(\mathbb{T}) \to L^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$,
where $H^1_{\mathrm{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ is embedded into $\mathrm{Dom}(A)$.

Spectral instability result

Theorem (F. Natali, D.P., S. Wang, JNLW, 2025, in print)

The spectrum of $A : \mathrm{Dom}(A) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$ *completely covers the closed vertical strip given by*

$$\sigma(A) = \left\{ \lambda \in \mathbb{C} : \quad -\frac{\pi}{4} \leq \mathrm{Re}(\lambda) \leq \frac{\pi}{4} \right\}.$$

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In fact, we show

$$\sigma_p(A) = \left\{ \lambda \in \mathbb{C} : \quad -\frac{\pi}{4} < \mathrm{Re}(\lambda) < \frac{\pi}{4} \right\},$$

$$\rho(A) = \left\{ \lambda \in \mathbb{C} : \quad |\mathrm{Re}(\lambda)| > \frac{\pi}{4} \right\},$$

so that $\sigma_c(A) = \sigma(A) \backslash \sigma_p(A)$.

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To find $\sigma_p(A)$, we analyze $Af = \lambda f$, $f \in \mathrm{Dom}(A)$:

$$\frac{1}{4} x(2\pi - x) f'(x) + \frac{1}{4\pi} \int_0^{2\pi} (\pi - x) f(x) dx + \frac{1}{2} \Pi_0 \partial_x^{-1} \Pi_0 f = \lambda f(x),$$

with the constraint $\lambda \int_0^{2\pi} f(x) dx = 0$.

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Since $f \in C^\infty(0, 2\pi)$, the spectral problem is the ODE:

$$\frac{1}{4}x(2\pi - x)f''(x) + \frac{1}{2}(\pi - x)f'(x) + \frac{1}{2}f(x) - \frac{1}{4\pi}\int_0^{2\pi} f(x)dx = \lambda f'(x),$$

with two solutions $f_1(x) = 2\lambda - \pi + x$ and

$$f_2(x) \sim x^{\frac{2\lambda}{\pi}}, \quad x \to 0^+, \quad f_2(x) \sim (2\pi - x)^{-\frac{2\lambda}{\pi}}, \quad x \to (2\pi)^-.$$

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For

$$\sigma_p(A) = \left\{ \lambda \in \mathbb{C} : \quad -\frac{\pi}{4} < \mathrm{Re}(\lambda) < \frac{\pi}{4} \right\},$$

both $f_1, f_2 \in \mathrm{Dom}(A)$ and $c_1 f_1(x) + c_2 f_2(x)$ satisfies the constraint $\lambda \int_0^{2\pi} f(x) dx = 0$.

Nonlinear instability result

Theorem (F. Natali, D.P., S. Wang, JNLW, 2025, in print)

For every $\delta > 0$ *there exists* $\zeta_0 \in H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ *satisfying*

$$\|\zeta_0\|_{H^1_{\text{per}}} \leq \delta^2, \quad \|\zeta_0\|_{W^{1,\infty}} \leq \delta,$$

such that the unique local solution ζ *satisfies* $\|\zeta(t_0, \cdot)\|_{W^{1,\infty}} = 1$.

Nonlinear instability result

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For every $\delta > 0$ there exists $\zeta_0 \in H^1_{\rm per}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ satisfying

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such that the unique local solution ζ satisfies $\|\zeta(t_0,\cdot)\|_{W^{1,\infty}} = 1$.

The nonlinear evolution equation is

$$2c_* \partial_t \zeta = (c_*^2 - 2\eta_*)\partial_x \zeta - 2(\zeta - \zeta|_{x=0})\partial_x \zeta - \frac{1}{\pi}\langle \eta_*', \zeta\rangle + \frac{1}{2}\Pi_0 \partial_x^{-1} \Pi_0 \left[\zeta + 2(\partial_x \zeta)^2\right].$$

with two conserved quantities

$$\oint \zeta dx \quad \text{and} \quad \zeta|_{x=0} + \frac{1}{\pi}\oint (\partial_x \zeta)^2 dx = C_0.$$

Nonlinear instability result

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such that the unique local solution ζ *satisfies* $\|\zeta(t_0,\cdot)\|_{W^{1,\infty}} = 1$*.*

Characteristic curves for $x = X(t,s)$:

$$\begin{cases} 2c_*\partial_t X(t,s) = -(c_*^2 - 2\eta_*(X)) + 2(\zeta(t,X) - \zeta(t,0)), \\ X(0,s) = s. \end{cases}$$

and evolution of $Z(t,s) := \zeta(t, X(t,s))$ along the curves

$$\begin{cases} 2c_*\partial_t Z(t,s) = -\frac{1}{\pi}\langle \eta_*', \zeta\rangle + \frac{1}{2}\Pi_0\partial_x^{-1}\Pi_0(\zeta + 2(\partial_x\zeta)^2), \\ Z(0,s) = \zeta_0(s), \end{cases}$$

Nonlinear instability result

Theorem (F. Natali, D.P., S. Wang, JNLW, 2025, in print)

For every $\delta > 0$ there exists $\zeta_0 \in H^1_{\rm per}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ satisfying

$$\|\zeta_0\|_{H^1_{\rm per}} \leq \delta^2, \quad \|\zeta_0\|_{W^{1,\infty}} \leq \delta,$$

such that the unique local solution ζ satisfies $\|\zeta(t_0,\cdot)\|_{W^{1,\infty}} = 1$.

Assuming $\zeta_0 \in C^1(0,2\pi)$, we get for $V(t,s) := \partial_x \zeta(t, X(t,s))$:

$$\left\{ \begin{array}{l} 2c_* \partial_t V(t,s) = -2\eta_*'(X) V - V^2 + \frac{1}{2}(Z(t,s) + Z(t,0)), \\ V(0,s) = \zeta_0'(s), \end{array} \right.$$

with the one-sided limit to the peak at $V_0(t) := \lim\limits_{s \to 0^+} V(t,s)$:

$$2c_* V_0'(t) = \frac{\pi}{2} V_0(t) - V_0^2(t) + Z(t,0) \leq \frac{\pi}{2} V_0(t) + C_0.$$

Nonlinear instability result

Theorem (F. Natali, D.P., S. Wang, JNLW, 2025, in print)

For every $\delta > 0$ *there exists* $\zeta_0 \in H^1_{\mathrm{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ *satisfying*

$$\|\zeta_0\|_{H^1_{\mathrm{per}}} \leq \delta^2, \quad \|\zeta_0\|_{W^{1,\infty}} \leq \delta,$$

such that the unique local solution ζ *satisfies* $\|\zeta(t_0,\cdot)\|_{W^{1,\infty}} = 1$.

This gives the instability in the $W^{1,\infty}$-norm:

$$V_0(t) \leq \left(V_0(0) + \frac{2}{\pi} C_0\right) e^{\frac{\pi t}{4c_*}}.$$

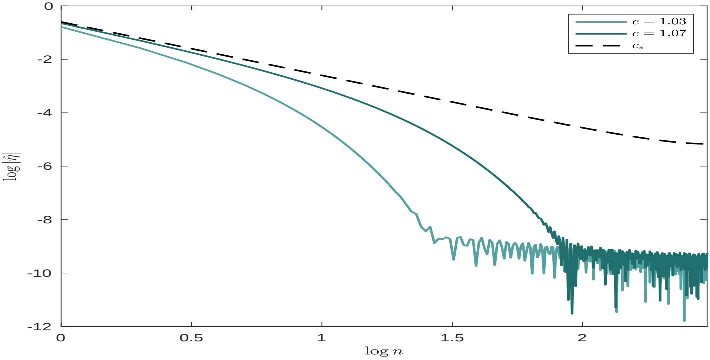
for $-\delta < V_0(0) < -\frac{2}{\pi}|C_0|$, where $|C_0| \lesssim \delta^2$.

5. Convergence of smooth waves to the peaked wave.

The wave profile is found from the first-order quadrature:

$$\left\{ \begin{array}{l} \left(\frac{d\eta}{dx}\right)^2 = \frac{2\mathcal{E} - \eta^2}{c^2 - 2\eta}, \\ \eta(\pm\pi) = -\sqrt{2\mathcal{E}}. \end{array} \right.$$

Coefficients of Fourier series decay exponentially for smooth waves and algebraically for the peaked wave.

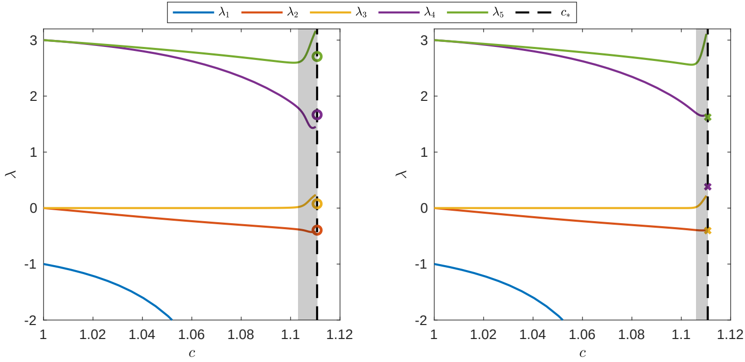


Eigenvalues of the self-adjoint operator.

The spectrum of $\mathcal{L} : H^2_{\rm per}(\mathbb{T}) \to L^2(\mathbb{T})$ is purely discrete:

$$\mathcal{L} = -\partial_x (c^2 - 2\eta)\partial_x + (2\eta'' - 1).$$

The lowest eigenvalue diverges in the limit of peaked waves and the numerical accuracy becomes poor.



Eigenfunctions of the self-adjoint operator.

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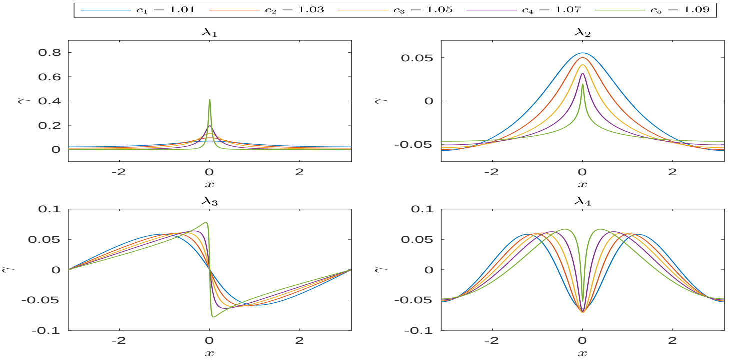
$$\mathcal{L} = -\partial_x (c^2 - 2\eta)\partial_x + (2\eta'' - 1).$$

Eigenfunctions become peaked in the limit of peaked waves.

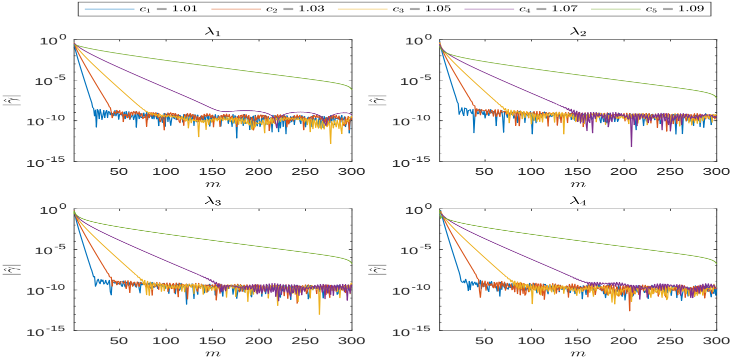


Eigenfunctions in the Fouier space

The spectrum of $\mathcal{L} : H^2_{\text{per}}(\mathbb{T}) \to L^2(\mathbb{T})$ is purely discrete:

$$\mathcal{L} = -\partial_x(c^2 - 2\eta)\partial_x + (2\eta'' - 1).$$

Coefficients decay slowly in the limit of peaked waves.

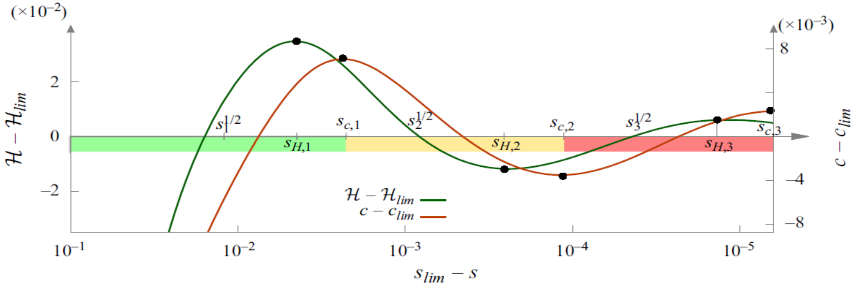


No convergence along the family of smooth waves?

Recall the Babenko equation for deep fluid

$$(c^2|\partial_x| - 1)\eta = \frac{1}{2}|\partial_x|\eta^2 + \eta|\partial_x|\eta,$$

with ∞-many oscillations for c in $(1, c_{\max})$ with $c_* \approx 1.0922$.



6. Summary

We considered the following model for $\eta = \eta(t,x)$:

$$\boxed{2c\partial_x\partial_t\eta = (c^2 - 2\eta)\partial_x^2\eta - (\partial_x\eta)^2 + \eta}$$

with x defined in the 2π-periodic domain $\mathbb{T}$ and $c > 0$ being a parameter for the wave speed.

- The smooth waves are linearly stable in the time evolution.
- The peaked wave is unstable in the time evolution.
- The cusped waves belong to H^1_{per} but do not belong to $H^1_{\text{per}} \cap W^{1,\infty}$, where local well-posedness is shown.