Instability of peaked waves

Dmitry Pelinovsky

Department of Mathematics, McMaster University, Canada
http://dmpeli.math.mcmaster.ca

Joint work with

Anna Geyer (Delft University of Technology, Netherlands)
Fabio Natali (University of Maringa, Brazil)
Long-wave models

The *Korteweg–De Vries equation* (1895) governs dynamics of small-amplitude long waves in a fluid:

\[ u_t + uu_x + \beta u_{xxx} = 0. \]

It arises from the dispersion relation for linear waves \( e^{i(kx-\omega t)} \):

\[
\omega^2 = c^2 k^2 + \beta k^4 + O(k^6) \quad \Rightarrow \quad \omega - ck = \frac{1}{2c} \beta k^3 + O(k^5).
\]
Long-wave models

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\]

The *Ostrovsky equation* (1978) models rotation effects:

\[
(u_t + uu_x + \beta u_{xxx})_x = \gamma^2 u,
\]

as follows from:

\[
\omega^2 = \gamma^2 + c^2 k^2 + \beta k^4 + \cdots \quad \Rightarrow \quad \omega - ck = \frac{\beta}{2c} k^3 + \frac{\gamma^2}{2ck} + \cdots
\]

As \( \beta \to 0 \), we obtain the *reduced Ostrovsky equation*. 
Long-wave models

The *Korteweg–De Vries equation* (1895) governs dynamics of small-amplitude long waves in a fluid:

\[ u_t + uu_x + \beta u_{xxx} = 0. \]

It arises from the dispersion relation for linear waves \( e^{i(kx - \omega t)} \):

\[ \omega^2 = c^2 k^2 + \beta k^4 + O(k^6) \Rightarrow \omega - ck = \frac{1}{2c} \beta k^3 + O(k^5). \]

The *Whitham equation* (1967) models full-dispersion effects:

\[ u_t + uu_x + K * u_x = 0, \]

where the Fourier transform of the convolution kernel:

\[ \hat{K}(k) = \sqrt{gh \frac{\tanh(kh)}{kh}} = \sqrt{gh} \left( 1 - \frac{1}{6} k^2 h^2 + \cdots \right) \]
Long-wave models

The *Korteweg–De Vries equation* (1895) governs dynamics of small-amplitude long waves in a fluid:

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\[ \omega^2 = c^2k^2 + \beta k^4 + O(k^6) \quad \Rightarrow \quad \omega - ck = \frac{1}{2c} \beta k^3 + O(k^5). \]

The *Camassa–Holm equation* (1994) models dispersion-modified nonlinear effects:

\[ u_t + 3uu_x - u_{txx} = 2u_xu_{xx} + uu_{xxx}. \]
Traveling wave solutions are solutions of the form

\[ u(x, t) = U(x - ct), \]

where \( z = x - ct \) is the travelling wave coordinate and \( c \) is the wave speed. For fixed \( c \), the wave profile \( U \) is either 2\( T \)-periodic or decaying to 0 at infinity.
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For the KdV equation, \( U \) satisfies

\[ \beta \frac{d^2 U}{dz^2} - cU + U^2 = 0. \]

All solutions are smooth.

[ODE textbooks]
Traveling wave solutions are solutions of the form

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where \( z = x - ct \) is the travelling wave coordinate and \( c \) is the wave speed. For fixed \( c \), the wave profile \( U \) is either \( 2T \)-periodic or decaying to 0 at infinity.

For the reduced Ostrovsky equation, \( U \) satisfies

\[ \frac{d}{dz} \left( (c - U) \frac{dU}{dz} \right) + U(z) = 0. \]

Solutions are smooth if \( c - U(z) > 0 \) for all \( z \).

[A.Geyer, D.P., 2017]
**Traveling wave solutions** are solutions of the form

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For the Whitham equation, \( U \) satisfies

\[ K \ast U = (c - U)U. \]

Solutions are smooth if \( c - U(z) > 0 \) for all \( z \).

[M. Ehrnström, H. Kalisch, 2013] [M. Ehrnström, E.Wahlén, 2015]
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where \( z = x - ct \) is the travelling wave coordinate and \( c \) is the wave speed. For fixed \( c \), the wave profile \( U \) is either 2\( T \)-periodic or decaying to 0 at infinity.

For the Camassa-Holm equation, \( U \) satisfies

\[ (c - U) \left[ \frac{d^2 U}{dz^2} - U \right] = 0. \]

All solutions are peaked with \( U(z_0) = c \) for some \( z_0 \in \mathbb{R} \).

[R. Camassa, D. Holm, J. Hyman, 1994]
KdV equation: smooth waves are linearly and orbitally stable

Reduced Ostrovsky equation: all smooth waves are linearly stable, but the limiting peaked wave is linearly unstable
[A.Geyer & D.P. 2019]

Whitham equation: small amplitude smooth waves are stable, but become unstable as they approach the peaked wave.
[J.Carter & H.Kalisch, 2014]

Camassa-Holm, Degasperis–Procesi, Novikov: peaked waves are orbitally and asymptotically stable
[A.Constantin & W.Strauss, 2000], [J.Lenells, 2005], [Z.Lin, Y .Liu, 2006], ...
Stability of smooth and peaked periodic waves

- KdV equation: smooth waves are linearly and orbitally stable

- Reduced Ostrovsky equation: all smooth waves are linearly stable, but the limiting \textit{peaked wave is linearly unstable}.
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Stability of smooth and peaked periodic waves

- **KdV equation**: smooth waves are linearly and orbitally stable

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- **Whitham equation**: small amplitude smooth waves are stable, but become unstable as they approach the peaked wave.
  [J. Carter & H. Kalisch, 2014]
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  [A.Constantin & W.Strauss, 2000], [J.Lenells, 2005], [Z.Lin, Y.Liu, 2006], ...
Plan of my talk

1. Instability of peaked waves in the reduced Ostrovsky equation

\[(u_t + uu_x)_x = u\]

- Cauchy problem in Sobolev spaces
- Existence of peaked periodic waves
- Linear instability of the peaked wave

2. Instability of peaked waves in the Camassa–Holm equation

\[u_t + 3uu_x - u_{txx} = 2u_xu_{xx} + uu_{xxx}\]

- Orbital stability of peakons in $H^1$
- Nonlinear instability of peakons beyond $H^1$
Consider Cauchy problem for the reduced Ostrovsky equation

\[
\begin{cases}
(u_t + uu_x)_x = u, \\
\left. u \right|_{t=0} = u_0.
\end{cases}
\]

- Local well-posedness for \( u_0 \in H^s \) with \( s > 3/2 \)
  
  [A.Stefanov et. al., 2010]

- Zero mass constraint is necessary in the periodic domain:
  \[
  \int_{-\pi}^{\pi} u_0(x) \, dx = 0.
  \]
Consider Cauchy problem for the reduced Ostrovsky equation

\[
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- Local well-posedness for \( u_0 \in H^s \) with \( s > 3/2 \)
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- Zero mass constraint is necessary in the periodic domain:
  \[
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  \]
- Local solutions break in finite time for large initial data.
  [Y.Liu & D.P. & A.Sakovich 2010]
- Global solutions exist for small initial data.
  [R.Grimshaw & D.P. 2014]
Global solutions for small initial data

<table>
<thead>
<tr>
<th>Theorem (R. Grimshaw &amp; D.P., 2014)</th>
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Let $u_0 \in H^3$ such that $1 - 3u_0''(x) > 0$ for all $x$. There exists a unique solution $u(t) \in C(\mathbb{R}, H^3)$ with $u(0) = u_0$. 

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Instability of peaked waves
Global solutions for small initial data

Theorem (R. Grimshaw & D.P., 2014)

Let \( u_0 \in H^3 \) such that \( 1 - 3u''_0(x) > 0 \) for all \( x \). There exists a unique solution \( u(t) \in C(\mathbb{R}, H^3) \) with \( u(0) = u_0 \).

The quantity \( 1 - 3u_{xx} \) appears in the Lax pair [A. Hone & M. Wang (2003)]

\[
\begin{align*}
3 \lambda \psi_{xxx} + (1 - 3u_{xx}) \psi &= 0, \\
\psi_t + \lambda \psi_{xx} + u \psi_x - u_x \psi &= 0,
\end{align*}
\]

and in the conserved quantities [J. Brunelli & S. Sakovich (2013)]

\[
\begin{align*}
E_0 &= \int_{\mathbb{R}} u^2 dx \\
E_1 &= \int_{\mathbb{R}} \left[ (1 - 3u_{xx})^{1/3} - 1 \right] dx, \\
E_2 &= \int_{\mathbb{R}} \frac{(u_{xxx})^2}{(1 - 3u_{xx})^{7/3}} dx
\end{align*}
\]
**Lemma**

Let $u_0 \in H^2_{\text{per}}$. The local solution $u \in C([0, T), H^2_{\text{per}})$ blows up in a finite time $T < \infty$ in the sense $\lim_{t \uparrow T} \|u(\cdot, t)\|_{H^2} = \infty$ if and only if

$$\liminf_{t \uparrow T} u_x(t, x) = -\infty,$$

while

$$\limsup_{t \uparrow T} |u(t, x)| < \infty.$$
Wave breaking for large initial data

Lemma

Let $u_0 \in H^2_{\text{per}}$. The local solution $u \in C([0, T), H^2_{\text{per}})$ blows up in a finite time $T < \infty$ in the sense $\lim_{t \uparrow T} \| u(\cdot, t) \|_{H^2_{\text{per}}} = \infty$ if and only if

$$\liminf_{t \uparrow T} u_x(t, x) = -\infty, \quad \text{while} \quad \limsup_{t \uparrow T} |u(t, x)| < \infty.$$

Theorem (J.Hunter, 1990)

Let $u_0 \in C^1_{\text{per}}$ and define

$$\inf_{x \in \mathbb{S}} u'_0(x) = -m \quad \text{and} \quad \sup_{x \in \mathbb{S}} |u_0(x)| = M.$$

If $m^3 > 4M(4 + m)$, a smooth solution $u(t, x)$ breaks in a finite time.
Lemma

Let \( u_0 \in H^2_{\text{per}} \). The local solution \( u \in C([0, T), H^2_{\text{per}}) \) blows up in a finite time \( T < \infty \) in the sense \( \lim_{t \uparrow T} \| u(\cdot, t) \|_{H^2} = \infty \) if and only if

\[
\lim \inf_{t \uparrow T} u_x(t, x) = -\infty, \quad \text{while} \quad \lim \sup_{t \uparrow T} |u(t, x)| < \infty.
\]

Theorem (Y. Liu, D.P. & A. Sakovich, 2010)

Assume that \( u_0 \in H^2_{\text{per}} \). The solution breaks if

either \( \int_S (u_0'(x))^3 \, dx < - \left( \frac{3}{2} \| u_0 \|_{L^2} \right)^{3/2} \),

or \( \exists x_0 : \quad u_0'(x_0) < - \left( \| u_0 \|_{L^\infty} + T_1 \| u_0 \|_{L^2} \right)^{1/2} \).
Wave breaking for large initial data

**Lemma**

Let $u_0 \in H^2_{\text{per}}$. The local solution $u \in C([0, T), H^2_{\text{per}})$ blows up in a finite time $T < \infty$ in the sense $\lim_{t \uparrow T} \|u(\cdot, t)\|_{H^2} = \infty$ if and only if

$$\lim_{t \uparrow T} \inf_{x} u_x(t, x) = -\infty, \quad \text{while} \quad \lim_{t \uparrow T} \sup_{x} |u(t, x)| < \infty.$$ 

**Conjecture on sharp wave breaking:**

Smooth solutions break in a finite time if $u_0 \in H^3$ yields sign-indefinite $1 - 3u_0''(x)$. 

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Let $c > 0$ and consider a periodic solution $U$ of

\[
\frac{d}{dz} \left( (c - U) \frac{dU}{dz} \right) + U = 0. \tag{ODE}
\]

The solution $U$ is smooth if and only if $(u, v) = (U, U')$ is a periodic orbit $\gamma_E$ of the planar system

\[
\begin{cases}
    u' = v, \\
    v' = \frac{-u + v^2}{c - u},
\end{cases}
\]

which has the first integral

\[
E(u, v) = \frac{1}{2} (c - u)^2 v^2 + \frac{c}{2} u^2 - \frac{1}{3} u^3.
\]

The solution $U$ is smooth if and only if $c - U(z) > 0$ for every $z$. 
Existence of smooth periodic waves

Let $c > 0$. The first integral is

$$E(u, v) = \frac{1}{2}(c - u)^2 v^2 + \frac{c}{2} u^2 - \frac{1}{3} u^3.$$ 

There exists a smooth family of periodic solutions parametrized by the energy $E \in (0, E_c)$, where $2T$ depends on $E$. 

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Peaked periodic wave

For $c = c_* := \pi^2/9$ there exists a solution with parabolic profile

$$U_*(z) := \frac{3z^2 - \pi^2}{18}, \quad z \in [-\pi, \pi],$$
Peaked periodic wave

For $c = c_* := \frac{\pi^2}{9}$ there exists a solution with parabolic profile

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which can be periodically continued.
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The peaked periodic wave $U_* \in H^s_{\text{per}}(-\pi, \pi)$ for $s < 3/2$:

$$U_*(z) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{3n^2} \cos(nz),$$

with $U_*(\pm \pi) = c_*$ and $U'_*(\pm \pi) = \pm \pi/3$. 
Peaked periodic wave

For \( c = c_* := \frac{\pi^2}{9} \) there exists a solution with parabolic profile

\[
U_*(z) := \frac{3z^2 - \pi^2}{18}, \quad z \in [-\pi, \pi],
\]

which can be periodically continued.

The peaked wave satisfies the border case:

\[
1 - 3U_*''(z) = 0 \text{ for } z \in (-\pi, \pi).
\]
Peaked periodic wave

For $c = c_* := \pi^2/9$ there exists a solution with parabolic profile

$$U_*(z) := \frac{3z^2 - \pi^2}{18}, \quad z \in [-\pi, \pi],$$

which can be periodically continued.

Theorem (A.Geyer & D.P, 2019)

The peaked periodic wave $U_*$ is the unique peaked solution with the jump at $z = \pm \pi$.

See also [Bruell & Dhara, 2019]
Spectral stability of smooth periodic waves

We consider *co-periodic* perturbations of the traveling waves, that is, *perturbations with the same period* $2T$. 

\[ u(t, x) = U(z) + v(z) e^{\lambda t}, \]

where $z = x - ct$. The spectral stability problem for a perturbation of the wave profile $U$ is given by

\[ \partial_z L v = \lambda v \]

with the self-adjoint linear operator

\[ L = P_0 (\partial_z^2 - 2z + c - U) P_0: \dot{L}^2_{\text{per}}(-T, T) \to \dot{L}^2_{\text{per}}(-T, T), \]

where $\dot{L}^2_{\text{per}}$ is the $L^2$ space of periodic functions with zero mean.

Definition

The travelling wave is *spectrally stable* with respect to co-periodic perturbations if the spectral problem

\[ \partial_z L v = \lambda v \]

with $v \in H^1_{\text{per}}(-T, T)$ has no eigenvalues $\lambda / \in i \mathbb{R}$. 

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Spectral stability of smooth periodic waves

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Using $u(t, x) = U(z) + v(z)e^{\lambda t}$, where $z = x - ct$, the spectral stability problem for a perturbation of the wave profile $U$ is given by

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where $\dot{L}_{\text{per}}^2$ is the $L^2$ space of periodic function with zero mean.

Definition

The travelling wave is spectrally stable with respect to co-periodic perturbations if the spectral problem $\partial_z Lv = \lambda v$ with $v \in H^1_{\text{per}}(-T, T)$ has no eigenvalues $\lambda \notin i\mathbb{R}$. 
Construct an augmented Lyapunov functional:

\[ F[u] := H[u] + cQ[u], \]

where

(energy) \[ H[u] = -\|\partial_x^{-1} u\|_{L_{per}^2}^2 - \frac{1}{3} \int_{-T}^{T} u^3 \, dx \]

(momentum) \[ Q[u] = \|u\|_{L_{per}^2}^2 \]
Spectral stability of smooth periodic waves

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A traveling wave \( U \) is a critical point of \( F[u] \), i.e. \( \delta F[U] = 0 \).
Spectral stability of smooth periodic waves

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A traveling wave \( U \) is a critical point of \( F[u] \), i.e. \( \delta F[U] = 0 \).

The Hessian of \( F[u] \) is the operator \( L \), i.e. \( \delta^2 F[U] v = \frac{1}{2} \langle Lv, v \rangle \).
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The Hessian of \( F[u] \) is the operator \( L \), i.e. \( \delta^2 F[U]v = \frac{1}{2} \langle Lv, v \rangle \).

\( L \) has exactly one negative eigenvalue, a simple zero eigenvalue with eigenvector \( \partial_z U \), and the rest of its spectrum is positive and bounded away from 0.
Theorem (Geyer & P., 2017)

The traveling wave $U$ is a local constrained minimizer of the energy $H[u]$ with fixed momentum $Q[u]$.
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The traveling wave $U$ is a local constrained minimizer of the energy $H[u]$ with fixed momentum $Q[u]$.

The constraint of fixed momentum $Q[u] := \|u\|_{L^2_{\text{per}}}^2 = q$ is equivalent to restricting the self-adjoint linear operator $L$ to the subspace $U^\perp = \left\{ v \in \dot{L}^2_{\text{per}}(-T, T) : \langle U, v \rangle_{L^2_{\text{per}}} = 0 \right\}$.
Theorem (Geyer & P., 2017)

The traveling wave $U$ is a local constrained minimizer of the energy $H[u]$ with fixed momentum $Q[u]$.

- The constraint of fixed momentum $Q[u] := \|u\|^2_{L^2_{\text{per}}} = q$ is equivalent to restricting the self-adjoint linear operator $L$ to the subspace

  $$U^\perp = \left\{ v \in \dot{L}^2_{\text{per}}(-T, T) : \langle U, v \rangle_{L^2_{\text{per}}} = 0 \right\}$$

- The operator $L|_{U^\perp}$ has a simple zero eigenvalue and a positive spectrum if

  $$\langle L^{-1}U, U \rangle = -\frac{d}{dc} \|U\|^2_{L^2_{\text{per}}(-T,T)} < 0.$$
Spectral stability of smooth periodic waves

**Theorem (Geyer & P., 2017)**

The traveling wave $U$ is a local constrained minimizer of the energy $H[u]$ with fixed momentum $Q[u]$.

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- By *Hamilton-Krein index theory* [Haragus & Kapitula, 08]

  $\#$ unstable EV of $\partial_z L \leq \#$ negative EV of $L|_{U^\perp}$
Theorem (Geyer & P., 2017)

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- By Hamilton-Krein index theory [Haragus & Kapitula, 08]

  \# unstable EV of $\partial_z L \leq$ \# negative EV of $L|_{U\perp}$

- The smooth periodic wave $U$ is spectrally stable in $L^2_{\text{per}}$.
For the peaked periodic wave $U_*$ travelling with the speed $c_*$, the spectral stability problem is

$$\partial_z L v = \lambda v$$

where the self-adjoint operator is

$$L = P_0 \left( \partial_z^{-2} + c_* - U_* \right) P_0 : \dot{L}_{\text{per}}^2 \rightarrow \dot{L}_{\text{per}}^2.$$
Spectral instability of the peaked periodic wave

For the peaked periodic wave $U_*$ travelling with the speed $c_*$, the spectral stability problem is

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**Lemma**

*The spectrum of the self-adjoint operator $L$ is $\sigma(L) = \{\lambda_-\} \cup \left[ 0, \frac{\pi^2}{6} \right].$*
Spectral instability of the peaked periodic wave

For the peaked periodic wave $U_*$ travelling with the speed $c_*$, the spectral stability problem is

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**Lemma**

The spectrum of the self-adjoint operator $L$ is $\sigma(L) = \{ \lambda_- \} \cup \left[ 0, \frac{\pi^2}{6} \right]$.

The spectral stability problem can not be solved by applying standard energy methods due to the lack of coercivity.
Spectral instability of the peaked periodic wave

For the peaked periodic wave $U_*$ travelling with the speed $c_*$, the spectral stability problem is

$$\partial_z L v = \lambda v$$

where the self-adjoint operator is

$$L = P_0 \left( \partial_z^{-2} + c_* - U_* \right) P_0 : \dot{L}^2_{\text{per}} \to \dot{L}^2_{\text{per}}.$$

Domain of $\partial_z L$ in $\dot{L}^2_{\text{per}}$ is larger than $H^1_{\text{per}}$:

$$\text{dom}(\partial_z L) = \{ v \in \dot{L}^2_{\text{per}} : \partial_z [(c_* - U_*)v] \in \dot{L}^2_{\text{per}} \}.$$

$U' \notin \dot{H}^1_{\text{per}}$, but $U' \in \text{dom}(\partial_z L)$ and $\partial_z LU' = 0$. 
Consider the linearized evolution for a co-periodic perturbation $v$ to the travelling wave $U$:

$$\begin{align*}
\begin{cases}
v_t + \partial_z [(U_*(z) - c_*)v] = \partial_z^{-1} v, & t > 0, \\
v|_{t=0} = v_0.
\end{cases}
\end{align*}$$

**Definition**

The travelling wave $U$ is **linearly unstable** if there exists $v_0 \in \text{dom}(\partial_z L)$ such that the unique global solution $v \in C(\mathbb{R}, \text{dom}(\partial_z L))$ satisfies $\lim_{t \to \infty} \|v(t)\|_{L^2} = \infty$. 

*Dmitry Pelinovsky, McMaster University*
Linear instability of the peaked periodic wave

Consider the linearized evolution for a co-periodic perturbation $v$ to the travelling wave $U$:

$$
\begin{cases}
    v_t + \partial_z [(U_\ast(z) - c_\ast)v] = \partial_z^{-1}v, & t > 0, \\
    v|_{t=0} = v_0.
\end{cases} \tag{linO}
$$

**Theorem (Geyer & P., 2019)**

The peaked travelling wave $U$ is linearly unstable with

$$
\|v(t)\|_{L^2} \geq C_0 e^{\pi t/6} \|v_0\|_{L^2}, \quad t > 0
$$

for some $C_0 > 0$. 

**Dmitry Pelinovsky, McMaster University**

Instability of peaked waves

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Linear instability of the peaked periodic wave

**Step 1:** The *truncated problem*

\[
\begin{aligned}
\begin{cases}
\nu_t + \frac{1}{6} \partial_z [(z^2 - \pi^2)\nu] = 0, & t > 0, \\
\nu|_{t=0} = \nu_0.
\end{cases}
\end{aligned}
\]  

(truncO)
Linear instability of the peaked periodic wave

▷ **Step 1**: The *truncated problem*

\[
\begin{aligned}
&v_t + \frac{1}{6} \partial_z \left[ (z^2 - \pi^2)v \right] = 0, \quad t > 0, \\
&v|_{t=0} = v_0.
\end{aligned}
\]

(TruncO)

**Method of characteristics.** The characteristic curves \( z = Z(s, t) \) are found explicitly and the solution of \( V(s, t) := v(Z(s, t), t) \) is

\[
V(s, t) = \frac{1}{\pi^2} [\pi \cosh(\pi t/6) - s \sinh(\pi t/6)]^2 v_0(s), \quad s \in [-\pi, \pi], \quad t \in \mathbb{R}.
\]
Step 1: The truncated problem

\[
\begin{aligned}
\begin{cases}
  v_t + \frac{1}{6} \partial_z \left[(z^2 - \pi^2)v\right] = 0, & t > 0, \\
  v|_{t=0} = v_0.
\end{cases}
\end{aligned}
\] (truncO)

Method of characteristics. The characteristic curves \( z = Z(s, t) \) are found explicitly and the solution of \( V(s, t) := v(Z(s, t), t) \) is

\[
V(s, t) = \frac{1}{\pi^2} \left[ \pi \cosh(\pi t/6) - s \sinh(\pi t/6) \right]^2 v_0(s), \quad s \in [-\pi, \pi], \quad t \in \mathbb{R}.
\]

This yields the linear instability result for the truncated problem:

Lemma

For every \( v_0 \in \text{dom}(\partial_z L) \) \( \exists! \) global solution \( v \in C(\mathbb{R}, \text{dom}(\partial_z L)) \). If \( v_0 \) is odd, then the global solution satisfies

\[
\frac{1}{2} \|v_0\|_{L^2} e^{\pi t/6} \leq \|v(t)\|_{L^2} \leq \|v_0\|_{L^2} e^{\pi t/6}, \quad t > 0.
\]
Linear instability of the peaked periodic wave

**Step 2:** The *full evolution problem*

\[
\begin{aligned}
\left\{ \begin{array}{l}
\nu_t + \frac{1}{6} \partial_z [(z^2 - \pi^2)\nu] = \partial_z^{-1} \nu, \\
\nu|_{t=0} = \nu_0.
\end{array} \right. \\
t > 0,
\end{aligned}
\]

A truncated problem \( \nu_t = A_0 \nu \) has a unique global solution in \( \dot{L}^2 \).

**Bounded Perturbation Theorem:**

\( A := A_0 + \partial_z^{-1} \) is the generator of \( C_0 \)-semigroup on \( \dot{L}^2 \).

**Lemma**

For every \( \nu_0 \in \text{dom}(\partial_z L) \), there exists a unique global solution \( \nu \in C(R, \text{dom}(\partial_z L)) \). If \( \nu_0 \) is odd, then the solution satisfies

\[
\|\nu_0\|_{L^2} e^{\pi t/6} \leq \|\nu(t)\|_{L^2} \leq \|\nu_0\|_{L^2} e^{\pi t/6},
\]
Step 2: The full evolution problem

\[
\begin{cases} 
    v_t + \frac{1}{6} \partial_z \left[ (z^2 - \pi^2) v \right] = \partial_z^{-1} v, & t > 0, \\
    v|_{t=0} = v_0. 
\end{cases}
\] (linO)

Generalized Meth. of Char. Treat \( \partial_z^{-1} v \) as a source term in (linO).

- truncated problem \( v_t = A_0 v \) has a unique global solution in \( \dot{L}^2_{\text{per}} \)
- Bounded Perturbation Theorem:
  
  \[ A := A_0 + \partial_z^{-1} \text{ is the generator of } C^0\text{-semigroup on } \dot{L}^2_{\text{per}} \]
Step 2: The full evolution problem

\[
\begin{cases}
v_t + \frac{1}{6} \partial_z \left[ (z^2 - \pi^2)v \right] = \partial_z^{-1} v, & t > 0, \\
v|_{t=0} = v_0.
\end{cases}
\] (linO)

Generalized Meth. of Char. Treat $\partial_z^{-1} v$ as a source term in (linO).

- truncated problem $v_t = A_0 v$ has a unique global solution in $\dot{L}^2_{\text{per}}$
- Bounded Perturbation Theorem:
  \[ A := A_0 + \partial_z^{-1} \text{ is the generator of } C^0\text{-semigroup on } \dot{L}^2_{\text{per}} \]

**Lemma**

For every $v_0 \in \text{dom}(\partial_z L)$ \(\exists!\) global solution $v \in C(\mathbb{R}, \text{dom}(\partial_z L))$. If $v_0$ is odd, then the solution satisfies

\[
C \|v_0\|_{L^2} e^{\pi t/6} \leq \|v(t)\|_{L^2} \leq \|v_0\|_{L^2} e^{\pi t/6}, \quad t > 0.
\]
Linear instability of the peaked periodic wave

**Step 2:** The full evolution problem

\[
\begin{align*}
\begin{cases}
v_t + \frac{1}{6} \partial_z \left[ (z^2 - \pi^2) v \right] = \partial_z^{-1} v, & t > 0, \\
v\big|_{t=0} = v_0.
\end{cases}
\end{align*}
\]

Generalized Meth. of Char. Treat \( \partial_z^{-1} v \) as a source term in (linO).

- truncated problem \( v_t = A_0 v \) has a unique global solution in \( \dot{L}^2_{\text{per}} \)
- Bounded Perturbation Theorem:
  
  \[ A := A_0 + \partial_z^{-1} \text{ is the generator of } C^0\text{-semigroup on } \dot{L}^2_{\text{per}} \]

**Lemma**

*For every* \( v_0 \in \text{dom}(\partial_z L) \) *exists! global solution* \( v \in C(\mathbb{R}, \text{dom}(\partial_z L)) \). *If* \( v_0 \) *is odd, then the solution satisfies*

\[
C \| v_0 \|_{L^2} e^{\pi t/6} \leq \| v(t) \|_{L^2} \leq \| v_0 \|_{L^2} e^{\pi t/6}, \quad t > 0.
\]

The peaked periodic wave is *linearly unstable.*
Spectral instability of the peaked periodic wave

Theorem (Geyer & P., 2020)

\[ \sigma(\partial_z L) = \left\{ \lambda \in \mathbb{C} : -\frac{\pi}{6} \leq \text{Re}(\lambda) \leq \frac{\pi}{6} \right\}, \]

where

\[ \partial_z L v := \partial_z [(c_* - U_*)v] + \partial_z^{-1} v \]

with

\[ \text{dom}(\partial_z L) = \left\{ v \in \dot{L}^2_{\text{per}} : \partial_z [(c_* - U_*)v] \in \dot{L}^2_{\text{per}} \right\}. \]

0 \in \sigma(\partial_z L) \text{ because } U'_* \in \text{dom}(\partial_z L) \text{ and } \partial_z L U'_* = 0.
Nonlinear instability ???

Consider Cauchy problem for *the reduced Ostrovsky equation*

\[
\begin{cases}
(u_t + uu_x)_x = u, \\
u|_{t=0} = u_0.
\end{cases}
\]

Does linear instability imply nonlinear instability?
Nonlinear instability

Consider Cauchy problem for the reduced Ostrovsky equation

\[
\begin{cases}
(u_t + uu_x)_x = u, \\
u|_{t=0} = u_0.
\end{cases}
\]

Does linear instability imply nonlinear instability?

- Lack of well-posedness results for \( u_0 \in H^s_{\text{per}} \) with \( s < 3/2 \).
- Lack of information on dynamics of peaked perturbations to the peaked periodic wave.
Plan of part II

1. Instability of peaked waves in the reduced Ostrovsky equation

\[(u_t + uu_x)_x = u\]

▷ Cauchy problem in Sobolev spaces
▷ Existence of peaked periodic waves
▷ Linear instability of the peaked wave

2. Instability of peaked waves in the Camassa–Holm equation

\[u_t + 3uu_x - u_{txx} = 2u_x u_{xx} + uu_{xxx}.\]

▷ Orbital stability of peakons in $H^1$
▷ Nonlinear instability of peakons beyond $H^1$
Cauchy problem in Sobolev spaces

Let \( \varphi(x) = e^{-|x|} \) be the Greens function satisfying \( (1 - \partial_x^2) \varphi = 2\delta \).

The Cauchy problem for the Camassa–Holm equation can be written in the convolution form:

\[
\begin{cases}
    u_t + uu_x + \frac{1}{2} \varphi' * (u^2 + \frac{1}{2} u_x^2) = 0, \\
    u|_{t=0} = u_0.
\end{cases}
\]

The quantity \( m := (1 - \partial_x^2)u \) is referred as the momentum density.
Cauchy problem in Sobolev spaces

Let $\varphi(x) = e^{-|x|}$ be the Greens function satisfying $(1 - \partial_x^2)\varphi = 2\delta$. The Cauchy problem for the Camassa–Holm equation can be written in the convolution form:

$$\begin{cases}
  u_t + uu_x + \frac{1}{2} \varphi' \ast (u^2 + \frac{1}{2}u_x^2) = 0, \\
  u|_{t=0} = u_0.
\end{cases}$$

The quantity $m := (1 - \partial_x^2)u$ is referred as the momentum density.

- Local well-posedness for $u_0 \in H^s$ with $s > 3/2$.
  [Y.Li-P.Olver (2000)] [Rodriguez (2001)]

- Local and global well-posedness for $u_0 \in H^3$ if $m_0 \geq 0$
  [A.Constantin (2000)]

- Wave breaking for $u_0 \in H^3$ if $\exists x_0: (x - x_0)m_0(x) \leq 0$.
  [A.Constantin, J. Escher (1998)]
Cauchy problem in Sobolev spaces

Let \( \varphi(x) = e^{-|x|} \) be the Greens function satisfying \((1 - \partial_x^2) \varphi = 2 \delta\). The Cauchy problem for the Camassa–Holm equation can be written in the convolution form:

\[
\begin{cases}
    u_t + uu_x + \frac{1}{2} \varphi' \ast (u^2 + \frac{1}{2} u_x^2) = 0, \\
    u|_{t=0} = u_0.
\end{cases}
\]

The quantity \( m := (1 - \partial_x^2) u \) is referred as the momentum density.

- Ill-posedness and norm inflation for \( u_0 \in H^s \) with \( s \leq 3/2 \).
  - [P. Byers (2006)] [Z. Guo et al. (2018)]
- Global existence of weak solutions \( u_0 \in H^1 \) with \( m_0 \geq 0 \).
  - [A. Constantin, L. Molinet (2000)]
- Global existence of weak solutions \( u_0 \in H^1 \).
Cauchy problem in Sobolev spaces

Let $\varphi(x) = e^{-|x|}$ be the Greens function satisfying $(1 - \partial_x^2)\varphi = 2\delta$.

The Cauchy problem for the Camassa–Holm equation can be written in the convolution form:

$$
\begin{cases}
  u_t + uu_x + \frac{1}{2} \varphi' \ast \left( u^2 + \frac{1}{2} u_x^2 \right) = 0, \\
  u|_{t=0} = u_0.
\end{cases}
$$

The quantity $m := (1 - \partial_x^2)u$ is referred as the momentum density.

- Uniqueness of weak global solutions $u_0 \in H^1$.
  
  [A. Bressan, G. Chen, Q. Zhang (2015)]

- Continuous dependence for $u_0 \in H^1 \cap W^{1,\infty}$ but no global existence in $H^1 \cap W^{1,\infty}$.
  
  [F. Linares, G. Ponce, and T. Sideris (2019)]

- Local solutions may break in a finite time with $u_x(t, x) \to -\infty$ at some $x \in \mathbb{R}$ as $t \nearrow T$.  

Dmitry Pelinovsky, McMaster University

Instability of peaked waves
Existence and stability of peakons

For every $c \in \mathbb{R}$, $u(t, x) = c \varphi(x - ct)$ is a solution to

$$u_t + uu_x + \frac{1}{2} \varphi' \ast \left( u^2 + \frac{1}{2} u_x^2 \right) = 0.$$
Existence and stability of peakons

For every $c \in \mathbb{R}$, $u(t, x) = c \varphi(x - ct)$ is a solution to

$$u_t + uu_x + \frac{1}{2} \varphi' * \left( u^2 + \frac{1}{2} u_x^2 \right) = 0.$$ 

There exist two conserved quantities:

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2) dx, \quad F(u) = \int_{\mathbb{R}} u(u^2 + u_x^2) dx.$$

such that $\|u(t, \cdot)\|_{H^1} = \|u_0\|_{H^1}$ for almost every $t \in \mathbb{R}$.

Theorem (A. Constantin–L.Molinet (2001))

$\varphi$ is a unique (up to translation) minimizer of $E(u)$ in $H^1$ subject to $3F(u) = 2E(u)$. Consequently, global solutions with $u_0 \in H^3$ with $m_0 \geq 0$ close to $\varphi$ in $H^1$ stay close to $\{\varphi(\cdot - a)\}_{a \in \mathbb{R}}$ in $H^1$ for all $t$. 
Existence and stability of peakons

For every $c \in \mathbb{R}$, $u(t, x) = c\varphi(x - ct)$ is a solution to

$$u_t + uu_x + \frac{1}{2} \varphi' \ast \left( u^2 + \frac{1}{2} u_x^2 \right) = 0.$$ 

Theorem (A. Constantin–W. Strauss (2000))

*For every small $\varepsilon > 0$, if the initial data satisfies* 

$$\|u_0 - \varphi\|_{H^1} < \left( \frac{\varepsilon}{3} \right)^4,$$

*then the solution satisfies* 

$$\|u(t, \cdot) - \varphi(\cdot - \xi(t))\|_{H^1} < \varepsilon, \quad t \in (0, T),$$

*where $\xi(t)$ is a point of maximum for $u(t, \cdot)$.*
Existence and stability of peakons

For every $c \in \mathbb{R}$, $u(t, x) = c \varphi(x - ct)$ is a solution to

$$u_t + uu_x + \frac{1}{2} \varphi' * \left( u^2 + \frac{1}{2} u_x^2 \right) = 0.$$

- Asymptotic stability of peakons for $u_0 \in H^1$ with $m_0 \geq 0$.
  [L. Molinet (2018)]

- Asymptotic stability of trains of peakons and anti-peakons.
  [L. Molinet (2019)]

- Inverse scattering for weak global solutions with multi-peakons.
Consider solutions of the Cauchy problem:

\[
\begin{aligned}
&u_t + uu_x + Q[u] = 0, \\
&u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty},
\end{aligned}
\]

where \( Q[u] := \frac{1}{2} \varphi' * (u^2 + \frac{1}{2} u_x^2) \). Moreover, assume that \( u_0 \) is piecewise \( C^1 \) with a single peak.
Instability of peakons

Consider solutions of the Cauchy problem:

\[
\begin{cases}
    u_t + uu_x + Q[u] = 0, \\
    u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty},
\end{cases}
\]

where \( Q[u] := \frac{1}{2} \varphi' * \left( u^2 + \frac{1}{2} u_x^2 \right) \). Moreover, assume that \( u_0 \) is piecewise \( C^1 \) with a single peak.

**Theorem (F. Natali–D.P. (2019))**

*For every \( \delta > 0 \), there exist \( t_0 > 0 \) and \( u_0 \in H^1 \cap W^{1,\infty} \) satisfying*

\[
\| u_0 - \varphi \|_{H^1} + \| u'_0 - \varphi' \|_{L^\infty} < \delta,
\]

*such that the global conservative solution satisfies*

\[
\| u_x(t_0, \cdot) - \varphi' (\cdot - \xi(t_0)) \|_{L^\infty} > 1,
\]

*where \( \xi(t) \) is a point of peak of \( u(t, \cdot) \) for \( t \in [0, t_0] \).*
Consider solutions of the Cauchy problem:

\[
\begin{align*}
  &u_t + uu_x + Q[u] = 0, \\
  &u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty},
\end{align*}
\]

where \( Q[u] := \frac{1}{2} \varphi' \ast (u^2 + \frac{1}{2}u_x^2) \). Moreover, assume that \( u_0 \) is piecewise \( C^1 \) with a single peak.

Weak formulation of the unique global conservative solution:

\[
\int_0^\infty \int_\mathbb{R} \left( u\psi_t + \frac{1}{2}u^2\psi_x - Q[u]\psi \right) \, dx \, dt + \int_\mathbb{R} u_0(x)\psi(0, x) \, dx = 0,
\]

where \( \psi \in C^1_c(\mathbb{R}^+ \times \mathbb{R}) \).
Consider solutions of the Cauchy problem:

\[
\begin{aligned}
&\left\{
\begin{array}{l}
  u_t + uu_x + Q[u] = 0, \\
  u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty},
\end{array}
\right.
\end{aligned}
\]

where \(Q[u] := \frac{1}{2} \varphi' \ast (u^2 + \frac{1}{2} u_x^2)\). Moreover, assume that \(u_0\) is piecewise \(C^1\) with a single peak.


- If \(u \in H^1(\mathbb{R})\), then \(Q[u] \in C(\mathbb{R})\).

- If \(u \in H^1(\mathbb{R}) \cap C^1(-\infty, 0) \cap C^1(0, \infty)\), then \(Q[u] \in C(\mathbb{R}) \cap C^1(-\infty, 0) \cap C^1(0, \infty)\).
Consider solutions of the Cauchy problem:

\[
\begin{aligned}
    u_t + uu_x + Q[u] &= 0, \\
    u|_{t=0} &= u_0 \in H^1 \cap W^{1,\infty},
\end{aligned}
\]

where \( Q[u] := \frac{1}{2} \varphi' \ast (u^2 + \frac{1}{2} u_x^2) \). Moreover, assume that \( u_0 \) is piecewise \( C^1 \) with a single peak.

If \( u(t, \cdot + \xi(t)) \in H^1(\mathbb{R}) \cap C^1(-\infty, 0) \cap C^1(0, \infty) \) for \( t \in (0, T) \) with \( \xi(t) \in C^1(0, T) \), then

\[
    \frac{d\xi}{dt} = u(t, \xi(t)), \quad t \in (0, T).
\]
Decomposition near a single peakon

Consider a decomposition:

\[ u(t, x) = \varphi(x - t - a(t)) + v(t, x - t - a(t)), \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}, \]

where \( a'(t) = v(t, 0) \). Then \( v(t, x) \) satisfies the Cauchy problem:

\[
\begin{cases}
    v_t = (1 - \varphi)v_x + \varphi w + (v|_{x=0} - v)v_x - Q[v], & t \in (0, T), \\
    v|_{t=0} = v_0,
\end{cases}
\]

where \( w(t, x) = \int_0^x v(t, y)dy \).
Decomposition near a single peakon

Consider a decomposition:

\[ u(t, x) = \varphi(x - t - a(t)) + v(t, x - t - a(t)), \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}, \]

where \( a'(t) = v(t, 0) \). Then \( v(t, x) \) satisfies the Cauchy problem:

\[
\begin{cases}
    v_t = (1 - \varphi)v_x + \varphi w + (v|_{x=0} - v)v_x - Q[v], & t \in (0, T), \\
    v|_{t=0} = v_0,
\end{cases}
\]

where \( w(t, x) = \int_0^x v(t, y)dy \).

The characteristic coordinates \( X(t, s) \) satisfies the IVP:

\[
\begin{cases}
    \frac{dX}{dt} = \varphi(X) - 1 + v(t, X) - v(t, 0), & t \in (0, T), \\
    X|_{t=0} = s,
\end{cases}
\]

which has a unique solution since \( \varphi \) and \( v \) is Lipschitz continuous.

\[ \Rightarrow X(t, 0) = 0 \] is invariant in \( t \).
Evolution near a single peakon

On characteristic curves, $V(t, s) := v(t, X(t, s))$ satisfies:

$$\begin{cases} \frac{dV}{dt} = \varphi(X)w(t, X) - Q[v](X), \\
V|_{t=0} = v_0(s).
\end{cases}$$

whereas $V'(t, s) := v_x(t, X(t, s))$ satisfies

$$\begin{cases} \frac{dV'}{dt} = -\varphi'(X)V' + \varphi(X)V + \varphi'(X)w(t, X) - \frac{1}{2}(V')^2 + V^2 - P[v](X), \\
V'|_{t=0} = v_0'(s).
\end{cases}$$

where $P[v](x) := \frac{1}{2} \int_{\mathbb{R}} \varphi(x - y) \left( [v(y)]^2 + \frac{1}{2} [v'(y)]^2 \right) dy$. 
Evolution near a single peakon

On characteristic curves, $V(t, s) := v(t, X(t, s))$ satisfies:

$$\begin{aligned}
\frac{dV}{dt} &= \varphi(X)w(t, X) - Q[v](X), \\
V\big|_{t=0} &= v_0(s).
\end{aligned}$$

whereas $V'(t, s) := v_x(t, X(t, s))$ satisfies

$$\begin{aligned}
\frac{dV'}{dt} &= -\varphi'(X)V' + \varphi(X)V + \varphi'(X)w(t, X) - \frac{1}{2}(V')^2 + V^2 - P[v](X), \\
V'|_{t=0} &= v'_0(s).
\end{aligned}$$

where $P[v](x) := \frac{1}{2} \int_{\mathbb{R}} \varphi(x - y) \left( [v(y)]^2 + \frac{1}{2} [v'(y)]^2 \right) dy$.

From one side of the peak, $V_0(t) = V(t, 0)$, $V'_0(t) = V'(t, +0)$:

$$\frac{d}{dt}(V_0 + V'_0) = (V_0 + V'_0) + V_0^2 - \frac{1}{2}(V'_0)^2 - Q[v](0) - P[v](0).$$
Evolution near a single peakon

On characteristic curves, $V(t, s) := v(t, X(t, s))$ satisfies:

$$
\begin{aligned}
\frac{dV}{dt} &= \varphi(X)w(t, X) - Q[v](X), \\
V\big|_{t=0} &= v_0(s).
\end{aligned}
$$

whereas $V'(t, s) := v_x(t, X(t, s))$ satisfies

$$
\begin{aligned}
\frac{dV'}{dt} &= -\varphi'(X)V' + \varphi(X)V + \varphi'(X)w(t, X) - \frac{1}{2}(V')^2 + V^2 - P[v](X), \\
V'|_{t=0} &= v'_0(s).
\end{aligned}
$$

where $P[v](x) := \frac{1}{2} \int_{\mathbb{R}} \varphi(x - y) \left( [v(y)]^2 + \frac{1}{2} [v'(y)]^2 \right) dy$.

Integrating with the integrating factors,

$$
\frac{d}{dt} \left[ e^{-t}(V_0 + V'_0) \right] = e^{-t} \left[ V_0^2 - \frac{1}{2} (V'_0)^2 - Q[v](0) - P[v](0) \right] \leq e^{-t}V_0^2.
$$
Evolution near a single peakon

On characteristic curves, $V(t, s) := v(t, X(t, s))$ satisfies:

\[
\begin{aligned}
\frac{dV}{dt} &= \varphi(X)w(t, X) - Q[v](X), \\
V|_{t=0} &= v_0(s).
\end{aligned}
\]

whereas $V'(t, s) := v_x(t, X(t, s))$ satisfies

\[
\begin{aligned}
\frac{dV'}{dt} &= -\varphi'(X)V' + \varphi(X)V + \varphi'(X)w(t, X) - \frac{1}{2}(V')^2 + V^2 - P[v](X), \\
V'|_{t=0} &= v'_0(s).
\end{aligned}
\]

where $P[v](x) := \frac{1}{2} \int_{\mathbb{R}} \varphi(x - y) ([v(y)]^2 + \frac{1}{2}[v'(y)]^2) \, dy$.

This yields the bound

\[
V_0(t) + V'_0(t) \leq e^t \left[ V_0(0) + V'_0(0) + \int_0^t e^{-\tau} V_0^2(\tau) \, d\tau \right], \quad t \in [0, T).
\]
Proof of instability

▷ From orbital stability in $H^1$ [A. Constant, W. Strauss (2000)]

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$$|V_0(t)| \leq \|v(t, \cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|v(t, \cdot)\|_{H^1} < \varepsilon.$$
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- Let $\lim_{x \to 0^+} v'_0(x) = -\|v'_0\|_{L^\infty} = -2\varepsilon^2$. If $v_0 \in H^1 \cap W^{1,\infty}$ satisfies $\|v_0\|_{L^\infty} + \|v'_0\|_{L^\infty} < \delta$, then $\forall \delta > 0$, $\exists \varepsilon > 0$ such that
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  \left(\frac{\varepsilon}{3}\right)^4 + 2\varepsilon^2 < \delta.
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- **Let** $\lim_{x \to 0^+} v_0'(x) = -\|v_0'\|_{L^\infty} = -2\varepsilon^2$. **If** $v_0 \in H^1 \cap W^{1,\infty}$

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- **From the bound above,** we have

  $$V_0(t) + V_0'(t) \leq -\varepsilon^2 e^t,$$

  hence $|V_0(t_0) + V_0'(t_0)| \geq 2$ for $t_0 := \log(2) - 2 \log(\varepsilon)$

  $$\Rightarrow |V_0'(t_0)| > 1.$$
Remarks

1. Instability of peakons with respect to peaked perturbations is consistent with local well-posedness for $u_0 \in H^1 \cap W^{1,\infty}$ and wave breaking in a finite time: $u_x(t, x) \to -\infty$ at some $x \in \mathbb{R}$.

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2. By means of characteristics, it follows that if $v_0 \in C^1(\mathbb{R})$, then $v(t, \cdot) \not\in C^1(\mathbb{R})$ for $t > 0$ because of the single peak at $x = \xi(t)$. 

Dmitry Pelinovsky, McMaster University
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3. Since $v_0(0) + v'_0(0) < 0$ for instability, the unstable solution actually breaks in a finite time [L. Brandolese (2014)].
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3. Since \( v_0(0) + v_0'(0) < 0 \) for instability, the unstable solution actually breaks in a finite time [L. Brandolese (2014)].

4. The same instability can be detected in the linearized equation

\[
\frac{d}{dt}(V_0 + V_0') = V_0 + V_0',
\]

from which \( V_0(t) + V_0'(t) = e^t [V_0(0) + V_0'(0)] \).
Consider the linearized equation at the single peakon:

\[
\begin{align*}
  v_t &= (1 - \varphi)v_x + \varphi w, \\
  v|_{t=0} &= v_0,
\end{align*}
\]

where \( w(t, x) = \int_0^x v(t, y)dy \).

**Theorem (F. Natali–D.P. (2019))**

For every \( v_0 \in H^1 \), there exists a unique global solution \( v \in C(\mathbb{R}, H^1) \) satisfying

\[
\|v(t, \cdot)\|_{H^1(0, \infty)}^2 = \|v_0\|_{H^1(0, \infty)}^2 + 2(e^t - 1) \int_0^\infty \varphi(s) \left( [v_0(s)]^2 + \frac{1}{2}[v'_0(s)]^2 \right) ds
\]

Linear instability does not imply nonlinear instability!
Global solutions and wave breaking in the Ostrovsky equation

\[(u_t + uu_x)_x = u.\]

- Smooth periodic waves are spectrally stable.
- Peaked wave is spectrally and linearly unstable.

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\[u_t + 3uu_x - u_{txx} = 2u_xu_{xx} + uu_{xxx}.\]

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Thank you! Questions ???