Rogue waves on the periodic background

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The rogue wave of the cubic NLS equation

The focusing nonlinear Schrödinger (NLS) equation

\[
i \psi_t + \frac{1}{2} \psi_{xx} + |\psi|^2 \psi = 0
\]

admits the exact solution

\[
\psi(x, t) = \left[1 - \frac{4(1 + 2it)}{1 + 4x^2 + 4t^2}\right] e^{it}.
\]

It was discovered by H. Peregrine (1983) and was labeled as the rogue wave.

Properties of the rogue wave:

- It is related to modulational instability of CW background \( \psi_0(x, t) = e^{it} \).
- It comes from nowhere: \( |\psi(x, t)| \to 1 \) as \( |x| + |t| \to \infty \).
- It is magnified at the center: \( M_0 := |\psi(0, 0)| = 3 \).
Possible developments:

- To generate higher-order rational solutions for multiple rogue waves...
- To extend constructions in other basic integrable PDEs...
Periodic wave background

The focusing nonlinear Schrödinger (NLS) equation

\[ i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0 \]

admits other wave solutions, e.g. the periodic waves of trivial phase

\[ \psi_{\text{dn}}(x, t) = \text{dn}(x; k)e^{i(1-k^2/2)t}, \quad \psi_{\text{cn}}(x, t) = k\text{cn}(x; k)e^{i(k^2-1/2)t} \]

where \( k \in (0, 1) \) is elliptic modulus.
Double-periodic wave background


\[ \psi(x, t) = k \frac{\text{cn}(t; k)\text{cn}(\sqrt{1+kx}; \kappa) + i\sqrt{1+k}\text{sn}(t; k)\text{dn}(\sqrt{1+kx}; \kappa)}{\sqrt{1+k}\text{dn}(\sqrt{1+kx}; \kappa) - \text{dn}(t; k)\text{cn}(\sqrt{1+kx}; \kappa)} e^{ikt}, \]

\[ \psi(x, t) = \frac{\text{dn}(t; k)\text{cn}(\sqrt{2x}; \kappa) + i\sqrt{k(1+k)}\text{sn}(t; k)}{\sqrt{1+k} - \sqrt{k}\text{cn}(t; k)\text{cn}(\sqrt{2x}; \kappa)} e^{ikt}, \quad \kappa = \frac{\sqrt{1-k}}{\sqrt{2}}. \]

where \( k \in (0, 1) \) is elliptic modulus.
Main question and background

Main question

Can we obtain a rogue wave on the background \( \psi_0 \) such that

\[
\inf_{x_0, t_0, \alpha_0 \in \mathbb{R}} \sup_{x \in \mathbb{R}} \left| \psi(x, t) - \psi_0(x - x_0, t - t_0)e^{i\alpha_0} \right| \to 0 \quad \text{as} \quad t \to \pm \infty
\]

This rogue wave *appears from nowhere and disappears without trace*.

Further questions:

- Magnification factors for rogue waves
- Spectral representation and inverse scattering
- Robustness (stability) in the time evolution.
- Extensions to quasi-periodic background.
- Extensions to multi-soliton background.
Darboux transformation as the main tool

Let $u$ be a solution of the NLS. It is a potential of the compatible Lax system

$$
\varphi_x = U(\lambda, u) \varphi,
$$

$$
U(\lambda, u) = \begin{pmatrix}
\lambda & u \\
-\bar{u} & -\lambda
\end{pmatrix}
$$

and

$$
\varphi_t = V(\lambda, u) \varphi,
$$

$$
V(\lambda, u) = i \begin{pmatrix}
\lambda^2 + \frac{1}{2}|u|^2 & \frac{1}{2} u_x + \lambda u \\
\frac{1}{2} \bar{u}_x - \lambda \bar{u} & -\lambda^2 - \frac{1}{2}|u|^2
\end{pmatrix},
$$

so that $\varphi_{xt} = \varphi_{tx}$.

Let $\varphi = (p_1, q_1)$ be a nonzero solution of the Lax system for $\lambda = \lambda_1 \in \mathbb{C}$. The following one-fold Darboux transformation (DT):

$$
\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1 \bar{q}_1}{|p_1|^2 + |q_1|^2},
$$

provides another solution $\hat{u}$ of the same NLS equation.
Preliminary literature

- Numerical computations of eigenfunctions for DT on $dn$, $cn$, and double-periodic backgrounds:
  (Kedziora–Ankiewicz–Akhmediev, 2014) (Calini–Schober, 2017)

- Emergence of rogue waves in simulations of modulation instability of $dn$-periodic waves:
  (Agafontsev–Zakharov, 2016)

- Magnification factors of quasi-periodic solutions from analysis of Riemann’s Theta functions:
  (Bertola–Tovbis, 2017) (Wright, 2019)

- Rogue waves from superpositions of nearly identical solitons:
  (Bilman–Buckingham, 2018) (Slunyaev, 2019)
Consider the spectral problem

\[ \varphi_x = U(\lambda, u)\varphi, \quad U(\lambda, u) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix} \]

Fix \( \lambda = \lambda_1 \in \mathbb{C} \) with \( \varphi = (p_1, q_1) \in \mathbb{C}^2 \) and set

\[
\begin{cases}
    u = p_1^2 + q_1^2, \\
    \bar{u} = \bar{p}_1^2 + \bar{q}_1^2.
\end{cases}
\]

The spectral problem becomes the Hamiltonian system of degree two generated by the Hamiltonian function

\[ H = \lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + \frac{1}{2} (p_1^2 + q_1^2)(\bar{p}_1^2 + \bar{q}_1^2). \]

The algebraic technique is called the “nonlinearization” of Lax pair (Cao–Geng, 1990) (Cao–Wu–Geng, 1999) (R. Zhou, 2009)
Hamiltonian system and constraints

The Hamiltonian system is integrable with two constants of motion:

\[ H = \lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + \frac{1}{2} (p_1^2 + q_1^2)(\bar{p}_1^2 + \bar{q}_1^2), \]

\[ F = i(p_1 q_1 - \bar{p}_1 \bar{q}_1). \]

The constraints between \( u \) and \((p_1, q_1)\) are extended as:

\[ u = p_1^2 + \bar{q}_1^2, \]

\[ \frac{du}{dx} + 2iFu = 2(\lambda_1 p_1^2 - \bar{\lambda}_1 \bar{q}_1^2), \]

\[ \frac{d^2u}{dx^2} + 2|u|^2u + 2iF \frac{du}{dx} - 4Hu = 4(\lambda_1^2 p_1^2 + \bar{\lambda}_1^2 \bar{q}_1^2). \]

Compatible potentials \( u(x) \) satisfy the closed second-order ODE:

\[ u'' + 2|u|^2u + 2icu' - 4bu = 0, \]

where \( c := F + i(\lambda_1 - \bar{\lambda}_1) \) and \( b := H + iF(\lambda_1 - \bar{\lambda}_1) + |\lambda_1|^2. \)
Integrability of the Hamiltonian system

The Hamiltonian system is a compatibility condition of the Lax equation

\[ \frac{d}{dx} W(\lambda) = U(\lambda, u) W(\lambda) - W(\lambda) U(\lambda, u), \]

where \( U(\lambda, u) \) is the same as in the Lax system and

\[ W(\lambda) = \begin{pmatrix} W_{11}(\lambda) & W_{12}(\lambda) \\ \bar{W}_{12}(-\lambda) & -\bar{W}_{11}(-\lambda) \end{pmatrix}, \]

with

\[ W_{11}(\lambda) = 1 - \frac{p_1 q_1}{\lambda - \lambda_1} + \frac{\bar{p}_1 \bar{q}_1}{\lambda + \bar{\lambda}_1}, \]

\[ W_{12}(\lambda) = \frac{p_1^2}{\lambda - \lambda_1} + \frac{\bar{q}_1^2}{\lambda + \bar{\lambda}_1}. \]

Simple algebra shows

\[ W_{11}(\lambda) = \frac{\lambda^2 + ic\lambda + b + \frac{1}{2}|u|^2}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)}, \quad W_{12}(\lambda) = \frac{u\lambda + icu + \frac{1}{2}u'}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)}. \]
Closure relations

The \((1, 2)\)-element of the Lax equation,

\[
\frac{d}{dx} W_{12}(\lambda) = 2\lambda W_{12}(\lambda) - 2uW_{11}(\lambda),
\]

yields the second-order equation on \(u\):

\[
u'' + 2|u|^2u + 2icu' - 4bu = 0.
\]

\(
\det W(\lambda)
\) is constant in \((x, t)\) and has simple poles at \(\lambda_1\) and \(-\bar{\lambda}_1\):

\[
\det[W(\lambda)] = -[W_{11}(\lambda)]^2 - W_{12}(\lambda)\bar{W}_{12}(-\lambda) = -\frac{P(\lambda)}{(\lambda - \lambda_1)^2(\lambda + \bar{\lambda}_1)^2}
\]

so that \(P(\lambda)\) is constant in \((x, t)\) and has roots at \(\lambda_1\) and \(-\bar{\lambda}_1\):

\[
P(\lambda) = (\lambda^2 + ic\lambda + b + \frac{1}{2}|u|^2)^2 - (u\lambda + icu + \frac{1}{2}u')(\bar{u}\lambda + ic\bar{u} - \frac{1}{2}\bar{u}')
\]
Conserved quantities

The second-order equation on $u$

$$u'' + 2|u|^2 u + 2icu' - 4bu = 0$$

is now closed with the conserved quantities

$$\begin{align*}
i(u'\bar{u} - u\bar{u}') - 2c|u|^2 &= 4a, \\
|u'|^2 + |u|^4 + 4b|u|^2 &= 8d.
\end{align*}$$

These equations describe a general class of traveling wave solutions:

$$\psi(x, t) = u(x + ct)e^{-2ibt}.$$ 

The polynomial $P(\lambda)$ in $\det W(\lambda)$ is given by

$$P(\lambda) = \lambda^4 + 2ic\lambda^3 + (2b - c^2)\lambda^2 + 2i(a + bc)\lambda + b^2 - 2ac + 2d,$$

with roots at $\lambda_1$ and $-\bar{\lambda}_1$. (Another pair also exists.)
Periodic waves of trivial phase

For traveling wave solutions:
- $c = 0$ can be set without loss of generality.
- $a = 0$ is set for waves with trivial phase.

The real function $u(x)$ is determined by the quadrature:

$$
\left( \frac{du}{dx} \right)^2 + u^4 + 4bu^2 = 8d
$$

with two parameters $b, d$. Parameterizing $V(u) = u^4 + 4bu^2 - 8d$ by two pairs of roots:

$$
\begin{cases}
-4b = u_1^2 + u_2^2, \\
-8d = u_1^2 u_2^2
\end{cases}
$$

we get two families of traveling wave solutions:
- $0 < u_2 < u_1$: $u(x) = u_1 \text{dn}(u_1 x; k)$
- $u_2 = i\nu_2$: $u(x) = u_1 \text{cn}(\alpha x; k), \, \alpha = \sqrt{u_1^2 + \nu_2^2}$
Lax spectrum of $dn$-periodic waves

Polynomial $P(\lambda)$ simplifies in terms of the turning points $u_1, u_2$:

$$P(\lambda) = \lambda^4 - \frac{1}{2}(u_1^2 + u_2^2)\lambda^2 + \frac{1}{16}(u_1^2 - u_2^2)^2$$

with two pairs of roots

$$\lambda_{1}^{\pm} = \pm \frac{u_1 + u_2}{2}, \quad \lambda_{2}^{\pm} = \pm \frac{u_1 - u_2}{2}.$$
Algebraic method with one eigenvalue

Lax spectrum of \(cn\)-periodic waves

If \(u_2 = i\nu_2\), there is one quadruplet of roots:

\[
\lambda_1^\pm = \pm \frac{u_1 + i\nu_2}{2}, \quad \lambda_2^\pm = \pm \frac{u_1 - i\nu_2}{2}.
\]
Let $\varphi = (p_1, q_1)$ be a nonzero solution of the Lax system for $\lambda = \lambda_1 \in \mathbb{C}$. The one-fold Darboux transformation

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1 \bar{q}_1}{|p_1|^2 + |q_1|^2},$$

gives another solution $\hat{u}$ of the same NLS equation.

**Question: which value of $\lambda_1$ to use?**
Algebraic method with one eigenvalue

Algebraic method - Step 2

Evaluating the matrix elements at simple poles $\lambda_1$ and $-\bar{\lambda}_1$

$$W_{11}(\lambda) = 1 - \frac{p_1 q_1}{\lambda - \lambda_1} + \frac{\bar{p}_1 \bar{q}_1}{\lambda + \bar{\lambda}_1} = \frac{\lambda^2 + ic\lambda + b + \frac{1}{2}|u|^2}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)},$$

$$W_{12}(\lambda) = \frac{p_1^2}{\lambda - \lambda_1} + \frac{\bar{q}_1^2}{\lambda + \bar{\lambda}_1} = \frac{u\lambda + icu + \frac{1}{2}u'}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)},$$

we can derive the inverse relations between the potential $u$ and the squared eigenfunctions:

$$p_1^2 = \frac{1}{\lambda_1 + \bar{\lambda}_1} \left( \frac{1}{2} u' + icu + \lambda_1 u \right),$$

$$q_1^2 = \frac{1}{\lambda_1 + \bar{\lambda}_1} \left( -\frac{1}{2} u' + icu + \lambda_1 u \right),$$

$$p_1 q_1 = -\frac{1}{\lambda_1 + \bar{\lambda}_1} \left( b + \frac{1}{2}|u|^2 + i\lambda_1 c + \lambda_1^2 \right).$$

The eigenfunction $\varphi = (p_1, q_1)$ is periodic if $u$ is periodic.
Let us define the second solution $\varphi = (\hat{p}_1, \hat{q}_1)$ by

$$\hat{p}_1 = p_1 \phi_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1 \phi_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2},$$

such that $p_1 \hat{q}_1 - \hat{p}_1 q_1 = 2$ (Wronskian is constant). Then, scalar function $\phi_1(x, t)$ satisfies

$$\frac{\partial \phi_1}{\partial x} = -\frac{4(\lambda_1 + \bar{\lambda}_1)\bar{p}_1 \bar{q}_1}{(|p_1|^2 + |q_1|^2)^2}$$

and

$$\frac{\partial \phi_1}{\partial t} = -\frac{4i(\lambda_1^2 - \bar{\lambda}_1^2)\bar{p}_1 \bar{q}_1}{(|p_1|^2 + |q_1|^2)^2} + \frac{2i(\lambda_1 + \bar{\lambda}_1)(u\bar{p}_1^2 + \bar{u}\bar{q}_1^2)}{(|p_1|^2 + |q_1|^2)^2}.$$ 

The system is compatible as it is obtained from Lax equation.
Second solutions for periodic waves

For periodic waves with the trivial phase, variables are separated by

\[ u(x, t) = U(x)e^{-2ibt}, \quad p_1(x, t) = P_1(x)e^{-ibt}, \quad q_1(x, t) = Q_1(x)e^{ibt}, \]

where \( U \) is real, either \( U(x) = \text{dn}(x; k) \) or \( U(x) = k\text{cn}(x; k) \), whereas \( |p_1|^2 + |q_1|^2 = \text{dn}(x; k) \) in both cases.

Integrating linear equations for \( \phi_1(x, t) \) yields

\[ \phi_1(x, t) = 2x + 2i(1 \pm \sqrt{1 - k^2})t \pm 2\sqrt{1 - k^2} \int_0^x \frac{dy}{\text{dn}^2(y; k)} \]

and

\[ \phi_1(x, t) = 2k^2 \int_0^x \frac{\text{cn}^2(y; k)dy}{\text{dn}^2(y; k)} \pm 2ik\sqrt{1 - k^2} \int_0^x \frac{dy}{\text{dn}^2(y; k)} + 2ikt \]

from which it is obvious that \( |\phi_1| \to \infty \) as \( t \to \pm\infty \).
Algebraic method - Step 3

Rogue waves on the background $u$ are generated by the DT:

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)\hat{p}_1 \hat{q}_1}{|\hat{p}_1|^2 + |\hat{q}_1|^2},$$

where

$$\hat{p}_1 = p_1 \phi_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1 \phi_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2},$$

As $t \to \pm \infty$,

$$\hat{u}(x, t)|_{|\phi_1| \to \infty} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1 \bar{q}_1}{|p_1|^2 + |q_1|^2}$$

which is a translation of the periodic wave $u$, e.g.

$$\hat{u}(x, t)|_{|\phi_1| \to \infty} = \frac{\sqrt{1 - k^2}}{\text{dn}(x; k)} = \text{dn}(x + K(k); k)$$

or

$$\hat{u}(x, t)|_{|\phi_1| \to \infty} = -\frac{k\sqrt{1 - k^2}\text{sn}(x; k)}{\text{dn}(x; k)} = k\text{cn}(x + K(k); k).$$
Rogue waves on the background $u$ are generated by the DT:

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)\hat{p}_1 \hat{q}_1}{|\hat{p}_1|^2 + |\hat{q}_1|^2},$$

where

$$\hat{p}_1 = p_1 \phi_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1 \phi_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2},$$

At the center of the rogue wave,

$$\hat{u}(x, t)|_{\phi_1=0} = u - \frac{2(\lambda_1 + \bar{\lambda}_1)p_1 \bar{q}_1}{|p_1|^2 + |q_1|^2} = 2u - \tilde{u},$$

hence the magnification factor does not exceed *three* in the one-fold transformation.
Rogue wave on the $dn$-periodic wave

The $dn$-periodic wave is

$$u(x, t) = \text{dn}(x; k)e^{i(1-k^2/2)t}$$

The rogue wave for the larger eigenvalue $\lambda_1$ has the larger magnification:

$$M(k) = 2 + \sqrt{1 - k^2}, \quad k \in [0, 1].$$
Another rogue wave on the $dn$-periodic wave

The $dn$-periodic wave is

$$u(x, t) = \text{dn}(x; k)e^{i(1-k^2/2)t}$$

The rogue wave for the smaller eigenvalue $\lambda_1$ has the smaller magnification:

$$M(k) = 2 - \sqrt{1 - k^2}, \quad k \in [0, 1].$$
Rogue wave on the $cn$-periodic wave

The $cn$-periodic wave is

$$\psi_{cn}(x, t) = kcn(x; k)e^{i(k^2 - 1/2)t}$$

The rogue wave has the exact magnification factor:

$$M(k) = 2, \quad k \in [0, 1].$$
Rogue wave on the $cn$-periodic wave

The $cn$-periodic wave is

$$\psi_{cn}(x, t) = kcn(x; k)e^{i(k^2 - 1/2)t}$$

The rogue wave has the exact magnification factor:

$$M(k) = 2, \quad k \in [0, 1].$$
Relation to modulation instability of the periodic wave

If $\lambda$ belongs to the Lax spectrum and $P(\lambda)$ is the polynomial in

$$P(\lambda) = \lambda^4 - \frac{1}{2}(u_1^2 + u_2^2)\lambda^2 + \frac{1}{16}(u_1^2 - u_2^2)^2$$

then $\Gamma := \pm 2i \sqrt{P(\lambda)}$ is in the modulation instability spectrum. (Deconinck–Segal, 2017) (Deconinck–Upsal, 2019)
Relation to modulation instability of the periodic wave

Here is an example of the periodic wave with nontrivial phase

\[ u(x) = R(x) e^{i\Theta(x)} e^{2ibt} \]

with

\[ R(x) = \sqrt{\beta - k^2 \text{sn}^2(x; k)}, \quad \Theta(x) = -2e \int_0^x \frac{dx}{R(x)^2}. \]
Relation to modulation instability of the periodic wave

Here is an example of the periodic wave with nontrivial phase

\[ u(x) = R(x)e^{i\Theta(x)}e^{2ibt} \]

with

\[ R(x) = \sqrt{\beta - k^2 \text{sn}^2(x; k)}, \quad \Theta(x) = -2e \int_0^x \frac{dx}{R(x)^2}. \]
Towards the double-periodic background

Algebraic method with two eigenvalues

Fix $\lambda = \lambda_1 \in \mathbb{C}$ with $\varphi = (p_1, q_1) \in \mathbb{C}^2$ and $\lambda = \lambda_2 \in \mathbb{C}$ with $\varphi = (p_2, q_2) \in \mathbb{C}^2$ such that $\lambda_1 \neq \pm \lambda_2$ and $\lambda_1 \neq \pm \bar{\lambda}_2$. Set

$$u = p_1^2 + \bar{q}_1^2 + p_2^2 + \bar{q}_2^2.$$ 

The algebraic method produces the third-order equation

$$u''' + 6|u|^2u' + 2ic(u'' + 2|u|^2u) + 4bu' + 8iau = 0,$$

with three constants of motion:

$$
\begin{align*}
d + \frac{1}{2}b|u|^2 + \frac{i}{4}c(u'\bar{u} - u\bar{u}') + \frac{1}{8}(u\bar{u}'' + u''\bar{u} - |u'|^2 + 3|u|^4) &= 0, \\
2e - a|u|^2 - \frac{1}{4}c(|u'|^2 + |u|^4) + \frac{i}{8}(u''\bar{u}' - u'\bar{u}'') &= 0, \\
f - \frac{i}{2}a(u'\bar{u} - u\bar{u}') + \frac{1}{4}b(|u'|^2 + |u|^4) + \frac{1}{16}(|u'' + 2|u|^2u|^2 - (u'\bar{u} - u\bar{u}')^2) &= 0.
\end{align*}
$$

Eigenvalues $\lambda_1$ and $\lambda_2$ are found among three roots of the polynomial

$$P(\lambda) = \lambda^6 + 2ic\lambda^5 + (2b - c^2)\lambda^4 + 2i(a + bc)\lambda^3 + (b^2 - 2ac + 2d)\lambda^2 + 2i(e + ab + cd)\lambda + f + 2bd - 2ce - a^2.$$
Towards the double-periodic background

**Double-periodic solutions** (Akhmediev, Eleonskii, Kulagin, 1987) correspond to $c = a = e = 0$. The solution takes the explicit form:

$$u(x, t) = [Q(x, t) + i\delta(t)] e^{i\theta(t)}.$$  

where $Q(x, t)$ and $\delta(t)$ are found from the first-order quadratures:

$$\delta(t) = \frac{\sqrt{z_1 z_3} \text{sn}(\mu t; k)}{\sqrt{z_3 - z_1 \text{cn}^2(\mu t; k)}},$$

with $0 \leq z_1 \leq z_2 \leq z_3$ and

$$Q(x, t) = Q_4 + \frac{(Q_1 - Q_4)(Q_2 - Q_4)}{(Q_2 - Q_4) + (Q_1 - Q_2)\text{sn}^2(\nu x; \kappa)},$$

with $Q_4 \leq Q_3 \leq Q_2 \leq Q_1$.

By construction, $\pm \sqrt{z_1}, \pm \sqrt{z_2}, \pm \sqrt{z_3}$ are roots of $P(\lambda)$:

$$P(\lambda) = \lambda^6 + 2b\lambda^4 + (b^2 + 2d)\lambda^2 + f + 2bd.$$
Towards the double-periodic background

Lax spectrum and rogue waves

The double-periodic solution if \( z_{1,2,3} \) are real:

\[
\begin{align*}
  u(x, t) &= k \frac{\cn(t; k) \cn(\sqrt{1 + kx}; \kappa) + i \sqrt{1 + k} \sn(t; k) \dn(\sqrt{1 + kx}; \kappa)}{\sqrt{1 + k} \dn(\sqrt{1 + kx}; \kappa) - \dn(t; k) \cn(\sqrt{1 + kx}; \kappa)} e^{it},
\end{align*}
\]
Towards the double-periodic background

Lax spectrum and rogue waves

The double-periodic solution if $z_1$ is real and $z_{2,3}$ are complex:

$$u(x, t) = \frac{\text{dn}(t, k) \text{cn}(\sqrt{2}x; \kappa) + i \sqrt{k(1+k)} \text{sn}(t, k)}{\sqrt{1 + k - \sqrt{k} \text{cn}(t, k) \text{cn}(\sqrt{2}x; \kappa)}} \text{e}^{ikt}, \quad \kappa = \frac{\sqrt{1 - k}}{\sqrt{2}}.$$
Towards the double-periodic background

Lax spectrum and rogue waves

The double-periodic solution if $z_1$ is real and $z_{2,3}$ are complex:

$$u(x, t) = \frac{\text{dn}(t; k)\text{cn}(\sqrt{2}x; \kappa) + i\sqrt{k(1 + k)}\text{sn}(t; k)}{\sqrt{1 + k - \sqrt{k}\text{cn}(t; k)\text{cn}(\sqrt{2}x; \kappa)}} e^{ikt}, \quad \kappa = \frac{\sqrt{1 - k}}{\sqrt{2}}.$$
Summary:

- New method is developed for computations of eigenvalues and eigenfunctions of the Lax system for periodic and double-periodic waves.

- New exact solutions are obtained for rogue waves on the background of periodic and double-periodic waves.

- Magnification factor is computed exactly at the rogue waves.

Further directions:

- Characterize eigenvalues, eigenfunctions, and rogue waves on general quasi-periodic solutions.

- Observe rogue waves on the periodic background in water wave experiments (Amin Chabchoub, Sydney).

Thank you! Questions???
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