Stability of water waves via pseudo-differential Babenko-type equations

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Outline of the talk

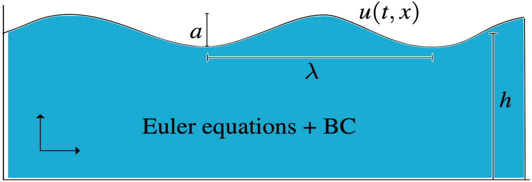
1. Stability patterns of the surface water waves: numerical results
2. Advantages of the pseudo-differential Babenko-type equations
3. Existence of traveling waves with smooth and peaked profiles
4. Stability criterion for co-periodic perturbations based on energy
5. Bifurcations of the modulational stability bands via normal forms

The talk is based on the joint work with

- Spencer Locke (PhD at University of Michigan),
- Sergey Dyachenko (University of Buffalo),
- Robert Marangell (University of Sydney).

1. Stability patterns of the surface water waves

Stokes waves are traveling waves of the permanent form in the irrotational motion of an incompressible fluid:



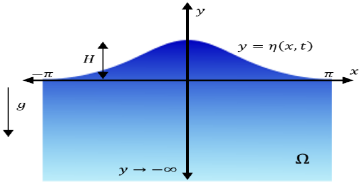
Traveling waves of small amplitudes were defined in

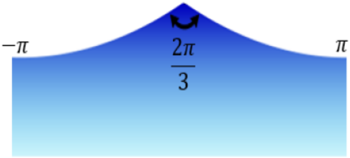
G. G. Stokes, On the theory of oscillatory waves, Transactions of the Cambridge Philosophical Society 8 (1847) 441

T. Levi-Civita, Determination rigoureuse des ondes permanentes dámpleur finite, Mathematische Annalen 93 (1925) 264

Stokes waves with the smooth and peaked profiles

In 1880, G. G. Stokes suggested existence of the limiting wave with the peaked profile in the family of waves with the smooth profiles:





Existence of peaked solutions was proven by J. Toland *et al.* and the $2\pi/3$-peaked singularity was proven by P. Plotnikov in 1982.

J. F. Toland, Proc. Roy. Soc. London 363 (1978) 469

C. J. Amick, L. E. Fraenkel, J. F. Toland, Acta Mathematica 148 (1982) 193

P. I. Plotnikov, Studies in Applied Mathematics 108 (2002) 217

Recent numerical results

More recently, numerical approximations of smooth periodic waves of large amplitudes were obtained with a high precision.

S. Dyachenko, P. Lushnikov, A. Korotkevich, Stud. Appl. Math. (2016)

S. Dyachenko, V. Hur, D. Silantyev, J. Fluid Mech. 955 (2023) A17

Spectral stability of smooth Stokes waves was explored numerically both for co-periodic and localized perturbations.

S. Dyachenko, A. Semenova, J. Comp. Phys. 492 (2023) 112411

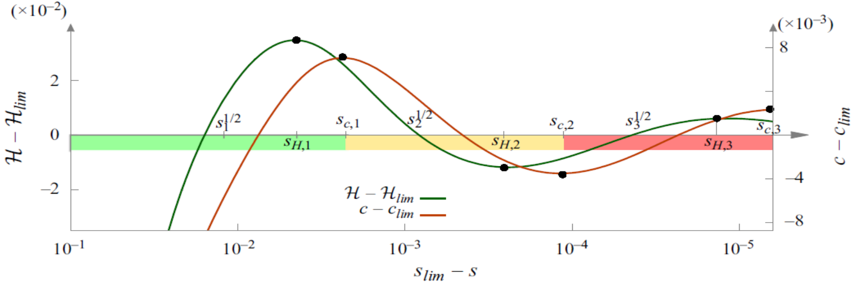
A. Korotkevich, P. Lushnikov, A. Semenova, S. Dyachenko, Stud.Appl.Math.(2023)

B. Deconinck, S. Dyachenko, A. Semenova, J. Fluid Mech. 995 (2024) A2

B. Deconinck, S. Dyachenko, P. Lushnikov, A. Semenova, Proc.Nat.Acad.Sci.(2023)

Recent numerical results

The wave energy $\mathcal{H}$ and the wave speed c oscillate as functions of the wave steepness s towards the limiting wave with the peaked profile:



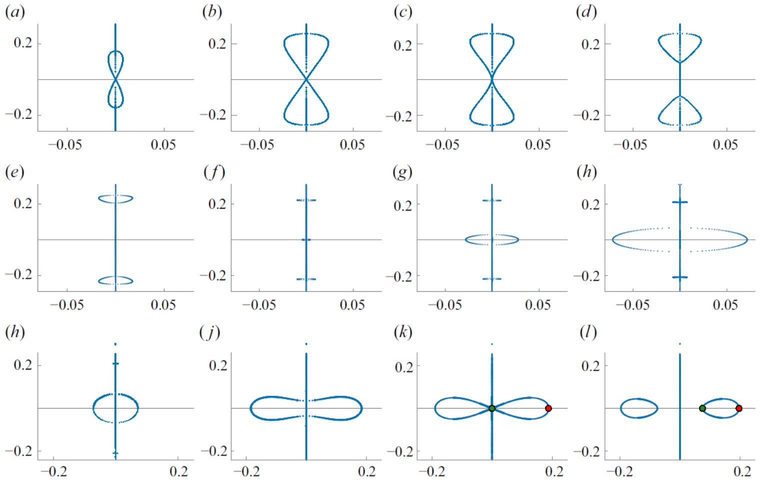
Instability of waves with respect to co-periodic perturbatios is related to the extremal points of energy.

M. Tanaka, J. Phys. Soc. Japan 52 (1983) 3047; J. Fluid Mech. 156 (1985) 281

P. G. Saffman, J. Fluid Mech. 159 (1985) 169

Recent numerical results

Instability bands with respect to arbitrary perturbations change in a sequence of bifurcations, which are repeated as steepness s increases.



Recent numerical results

Only the small-amplitude limit was considered by using rigorous mathematical analysis:

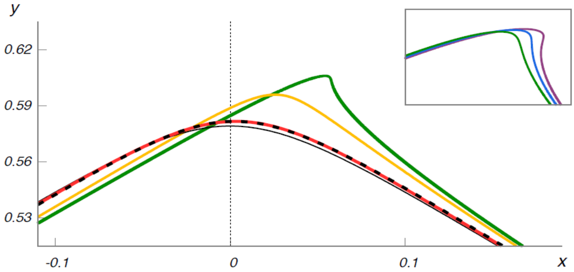
H. Q. Nguyen, W. A. Strauss, Comm. Pure Appl. Math. 76 (2023) 1035

M. Berti, A. Maspero, P. Ventura, Inventiones mathematicae 230 (2022) 651

M. Berti, A. Maspero, P. Ventura, Arch. Rational Mech. Anal. 247 (2023) 91

V. M. Hur, Z. Yang, Arch. Rational Mech. Anal. 247 (2023) 62

The wave profile becomes steep and breaks due to instability:



Wave breaking in reduced models

Instability of waves with the peaked profile due to wave breaking was well understood in many toy models of the fluid dynamics such as

- Camassa–Holm equation

$$u_t + u_x - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx}$$

F. Natali, D.E. Pelinovsky, J. Diff. Eqs. 268 (2020) 7342
A. Madiyeva, D.E. Pelinovsky, SIAM J. Math. Anal. 53 (2021) 3016

- Hunter–Saxton-type equation

$$2cu_{xt} = (c^2 - 2u)u_{xx} - (u_x)^2 + u$$

S. Locke, D. E. Pelinovsky, J. Fluid Mech. 1004 (2025) A1
F. Natali, D. E. Pelinovsky, S. Wang, J. Nonlinear Waves (2025) submitted

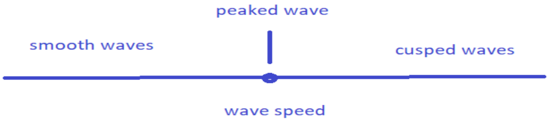
Wave breaking in reduced models

- Reduced Ostrovsky equation

$$(u_t + uu_x)_x = u$$

A. Geyer, D.E. Pelinovsky, SIAM J. Math. Anal. 51 (2019) 1188
A. Geyer, D.E. Pelinovsky, Proceedings of AMS 148 (2020) 5109

In all these models, the family of smooth waves remains linearly and nonlinearly stable all the way before the wave profile becomes peaked and the peaked wave becomes linearly and nonlinearly unstable.



Wave breaking in reduced models

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A. Geyer, D.E. Pelinovsky, SIAM J. Math. Anal. 51 (2019) 1188

A. Geyer, D.E. Pelinovsky, Proceedings of AMS 148 (2020) 5109

The main objective of this work is to develop the rigorous mathematical analysis of spectral instability of Stokes waves in full water wave equations for a deep fluid.

2. Advantages of the pseudo-differential formulation

- $\eta(x,t)$ - the free surface profile with the zero-mean constraint

$$\oint \eta dx = 0.$$

- $\phi(x,y,t)$ - velocity potential satisfying the Laplace equation in

$$D_\eta(t) := \{(x,y) : \quad -\pi \leq x \leq \pi, \quad -\infty < y \leq \eta(x,t)\}$$

- Periodic boundary conditions at $x = \pm\pi$.
- Decaying boundary condition $\varphi_y|_{y\to-\infty} = 0$.
- Nonlinear evolution equations at the free surface:

$$\left.\begin{aligned} \eta_t + \varphi_x \eta_x - \varphi_y = 0, \\ \varphi_t + \frac{1}{2}(\varphi_x)^2 + \frac{1}{2}(\varphi_y)^2 + \eta = 0, \end{aligned}\right\} \qquad \text{at } \; y = \eta(x,t),$$

2. Advantages of the pseudo-differential formulation

Three methods of study the dynamics in the Hamiltonian system

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1. Dirichlet-to-Neumann operator for $\varphi_y|_{y=\eta(x,t)}$ by
   W. Craig, C. Sulem, J. Comput. Phys. 108 (1993) 73

2. Reformulation in conformal variables by
   A. I. Dyachenko, E. A. Kuznetsov, V. E. Zakharov, Phys. Lett. A 221 (1996) 73

3. Reformulations with integral constraints by
   M. Ablowitz, A. Fokas, Z. Musslimani, J. Fluid Mech. 562 (2006) 313

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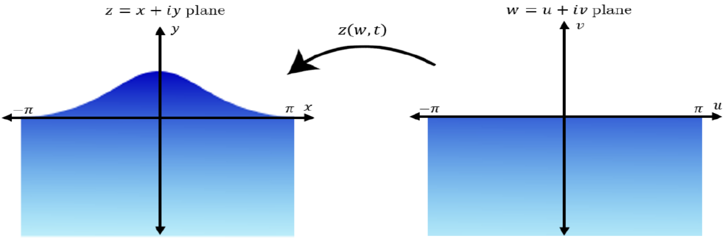
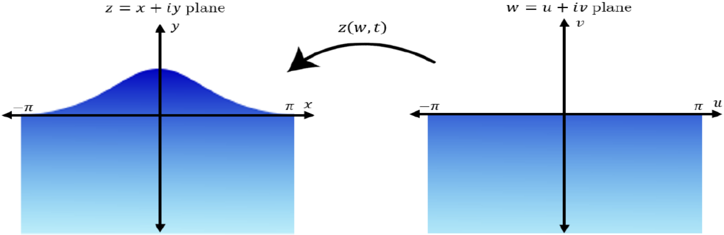
Conformal transformation



Conformal transformation

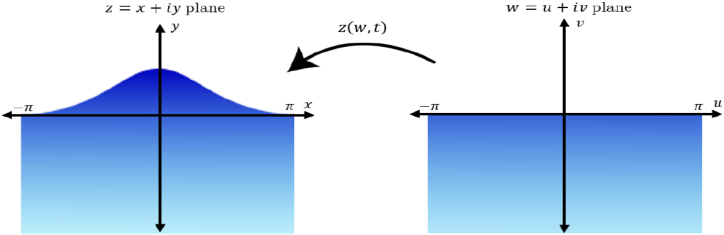


Cauchy–Riemann equations for $z = F(w,t)$ with holomorphic F in w:

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \qquad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}$$

in the fixed domain $\mathcal{D} := \{(u,v) : \ -\pi \leq u \leq \pi, \ -\infty \leq v \leq 0\}$ s.t. decaying condition as $v \to -\infty$ and periodic conditions at $u = \pm\pi$.

Conformal transformation



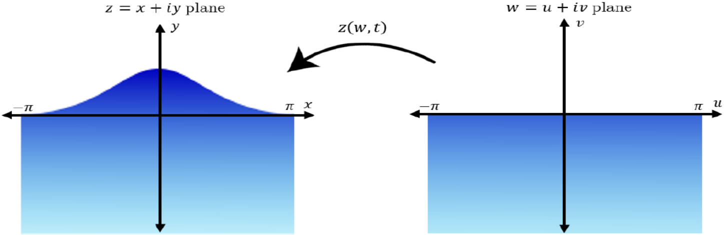
Fourier series solution:

$$x(u,v,t) = u + \sum_{n \in \mathbb{Z}} \hat{x}_n(t) e^{inu} \; e^{|n|v},$$

$$y(u,v,t) = v + \hat{y}_0(t) + \sum_{n \in \mathbb{Z}} \hat{x}_n(t) e^{inu} \; i \, \mathrm{sgn}(n) e^{|n|v},$$

where $\hat{y}_0(t)$ does not affect equations of motion.

Conformal transformation



Similarly, the velocity potential is uniquely represented by

$$\varphi(u,v,t) = \sum_{n\in\mathbb{Z}} \hat{\phi}_n(t) e^{inu}\, e^{|n|v},$$

where $\hat{\phi}_n(t)$ is the Fourier coefficient for $\phi(u,t) = \varphi(u,v=0,t)$. The other variables are $\xi(u,t) = x(u,v=0,t)$ and $\eta(u,t) = y(u,v=0,t)$.

A closed system of evolution equations

It follows from Cauchy–Riemann equations for $z = F(w,t)$ that

$$\eta = \hat{y}_0 + \mathcal{H}(\xi - u),$$

where $\mathcal{H}$ is Hilbert transform with Fourier symbol $\widehat{\mathcal{H}}_n = i\text{sgn}(n)$.

The closed system of evolution equations is obtained by the least action principle from the Lagrangian

$$\mathcal{L}(\phi,\xi,\eta) := \oint \phi(\eta_t \xi_u - \eta_u \xi_t) du - \frac{1}{2}\oint \phi \mathcal{K}\phi du - \frac{1}{2}\oint \eta^2 \xi_u du + \oint f(\eta - \hat{y}_0 - \mathcal{H}(\xi - u)) du,$$

where $\mathcal{K} = -\mathcal{H}\partial_u$ is a positive self-adjoint operator in $L^2(\mathbb{T})$ with Fourier symbol $\hat{\mathcal{K}}_n = |n|$ and f is the Lagrange multiplier to preserve the relation $\eta = \hat{y}_0 + \mathcal{H}(\xi - u)$.

A closed system of evolution equations

Using Euler–Lagrange equations in (ϕ, η, ξ), we obtain

$$\left\{ \begin{array}{l} \eta_t \xi_u - \eta_u \xi_t + \mathcal{H}\phi_u = 0, \\ -\phi_t \xi_u + \phi_u \xi_t - \eta \xi_u + f = 0, \\ \phi_t \eta_u - \phi_u \eta_t + \eta \eta_u + \mathcal{H} f = 0, \end{array} \right.$$

where $\xi_u = 1 + \mathcal{K}\eta$ and $\xi_t = \mathcal{H}\eta_t$.

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Using Euler–Lagrange equations in (ϕ, η, ξ), we obtain

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where $\xi_u = 1 + \mathcal{K}\eta$ and $\xi_t = \mathcal{H}\eta_t$.

Eliminating f yields the closed system of two evolution equations

$$\left\{ \begin{array}{l} (1 + \mathcal{K}\eta)\eta_t + \eta_u \mathcal{H}\eta_t + \mathcal{H}\phi_u = 0, \\ \phi_t \eta_u - \phi_u \eta_t + \eta \eta_u + \mathcal{H}\left[(1 + \mathcal{K}\eta)\phi_t + \phi_u \mathcal{H}\eta_t + (1 + \mathcal{K}\eta)\eta\right] = 0, \end{array} \right.$$

which replaces a full system of equations of motion for water waves.

The system admits a Hamiltonian formulation in canonical variables.

A. I. Dyachenko, E. A. Kuznetsov, V. E. Zakharov, Phys. Lett. A 221 (1996) 73

A closed system of evolution equations

Using Euler–Lagrange equations in (ϕ, η, ξ), we obtain

$$\begin{cases} \eta_t\xi_u - \eta_u\xi_t + \mathcal{H}\phi_u = 0, \\ -\phi_t\xi_u + \phi_u\xi_t - \eta\xi_u + f = 0, \\ \phi_t\eta_u - \phi_u\eta_t + \eta\eta_u + \mathcal{H}f = 0, \end{cases}$$

where $\xi_u = 1 + \mathcal{K}\eta$ and $\xi_t = \mathcal{H}\eta_t$.

The constraint $\oint \eta dx = 0$ becomes $\oint \eta\xi_u du = \oint \eta(1 + \mathcal{K}\eta)du = 0$. Additional constants of motion are

$$\oint \phi(1 + \mathcal{K}\eta)du, \quad \oint \phi\eta_u du, \quad \oint \left[\eta^2(1 + \mathcal{K}\eta) + \phi\mathcal{K}\phi\right] du.$$

These are two (vertical and horizontal) components of the momentum and the energy.

T. Benjamin, P. Olver, J. Fluid Mech. 125 (1982) 137

Babenko's equation

Traveling waves $\eta(u,t) = \eta(u - ct)$ satisfy $\phi_u = c\mathcal{K}\eta$, where the profile η is a solution of a scalar pseudo–differential equation

$$\boxed{(c^2\mathcal{K} - 1)\eta = \frac{1}{2}\mathcal{K}\eta^2 + \eta\mathcal{K}\eta.}$$

Existence of traveling waves with both smooth and peaked profiles is defined by solutions of Babenko's equation.

K. Babenko, Russian Academy of Sciences 294 (1987) 1033

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K. Babenko, Russian Academy of Sciences 294 (1987) 1033

Linear stability of traveling waves is defined by the linearization of the two equations near the traveling wave profile:

$$\left\{ \begin{array}{l} \eta(u,t) = \eta(u-ct) + v(u-ct,t), \\ \phi(u,t) = -c\mathcal{H}\eta(u-ct) + w(u-ct,t). \end{array} \right.$$

S. Dyachenko, A. Semenova, J. Comp. Phys. 492 (2023) 112411

Babenko's equation

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$$\boxed{(c^2\mathcal{K} - 1)\eta = \frac{1}{2}\mathcal{K}\eta^2 + \eta\mathcal{K}\eta.}$$

Linearization at (v, w) yields the closed system of linear equations

$$\left\{ \begin{array}{ll} \mathcal{M} v_t & = \mathcal{K} w, \\ \mathcal{M}^* w_t - 2c\mathcal{H} v_t & = \mathcal{L} v, \end{array} \right.$$

where $\mathcal{M} = 1 + \mathcal{K}\eta + \eta'\mathcal{H}$, $\mathcal{M}^* = 1 + \mathcal{K}\eta - \mathcal{H}(\eta' \cdot)$, and

$$\mathcal{L} = c^2\mathcal{K} - (1 + \mathcal{K}\eta) - \eta\mathcal{K} - \mathcal{K}(\eta \cdot),$$

where $\mathcal{L}$ is a self-adjoint linearized Babenko operator in $L^2(\mathbb{T})$.

Babenko's equation

Traveling waves $\eta(u,t) = \eta(u - ct)$ satisfy $\phi_u = c\mathcal{K}\eta$, where the profile η is a solution of a scalar pseudo–differential equation

$$\boxed{(c^2\mathcal{K} - 1)\eta = \frac{1}{2}\mathcal{K}\eta^2 + \eta\mathcal{K}\eta.}$$

Separation of variables as $v(u - ct, t) = \hat{v}(u - ct)e^{\lambda t}$, $w(u - ct, t) = \hat{w}(u - ct)e^{\lambda t}$ yields the spectral stability problem

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

with the domain in $H^1_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T})$ for co-periodic perturbations. The left-side is a self-adjoint unbounded operator and the right-side is a non-self-adjoint bounded operator in $L^2(\mathbb{T}) \times L^2(\mathbb{T})$.

3. Existence of traveling waves in Babenko's equation

Babenko's equation:

$$\boxed{c^2 \mathcal{K}\eta - \eta = \frac{1}{2}\mathcal{K}\eta^2 + \eta\mathcal{K}\eta,}$$

where $\mathcal{K} = -\mathcal{H}\partial_u$ with Fourier symbol $\widehat{\mathcal{K}}_n = |n|$.

Q: Which of your favorite integrable equation does it resemble?

3. Existence of traveling waves in Babenko's equation

Babenko's equation:

$$\boxed{c^2\mathcal{K}\eta - \eta = \frac{1}{2}\mathcal{K}\eta^2 + \eta\mathcal{K}\eta,}$$

where $\mathcal{K} = -\mathcal{H}\partial_u$ with Fourier symbol $\widehat{\mathcal{K}}_n = |n|$.

Q: Which of your favorite integrable equation does it resemble?
A: Benjamin–Ono equation $\eta_t + 2\eta\eta_x + \mathcal{H}\eta_{xx} = 0$

Traveling waves $\eta(x,t) = \eta(x+ct)$ satisfy

$$\boxed{\mathcal{K}\eta - c\eta = \eta^2.}$$

This equation has a family of smooth solutions for every $c > 1$

$$\eta(x) = \frac{\sinh\gamma}{\cos x - \cosh\gamma}, \quad c = \coth\gamma, \quad \gamma \in (0,\infty).$$

3. Existence of traveling waves in Babenko's equation

Babenko's equation:

$$\boxed{c^2\mathcal{K}\eta - \eta = \frac{1}{2}\mathcal{K}\eta^2 + \eta\mathcal{K}\eta,}$$

where $\mathcal{K} = -\mathcal{H}\partial_u$ with Fourier symbol $\widehat{\mathcal{K}}_n = |n|$.

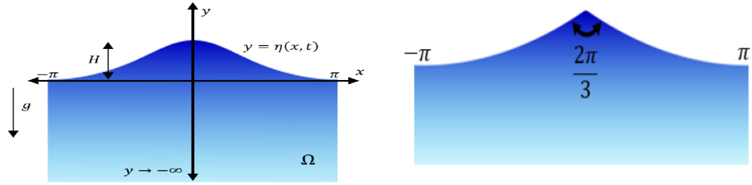
$c = 1$ is the local bifurcation point for nonzero 2π-periodic solutions.
Small-amplitude (Stokes) expansions are algorithmically computed:

$$\eta(u) = a\cos(u) + a^2\left[\cos(2u) - \frac{1}{2}\right] + \frac{3}{2}a^3\cos(3u) + \mathcal{O}(a^4)$$

where $a > 0$ is a small parameter for the wave amplitude which defines the speed $c^2 = 1 + a^2 + \mathcal{O}(a^4)$.
However, smooth waves of Babenko's equation exist only for $c \in (1, c_{\max})$ with $c_{\max} < \infty$.

Periodic solutions with the peaked profile

Recall that the physical profile of the surface elevation $(x, \eta(x))$ is given in the parametric form as $(\xi(u), \eta(u))$ with $\xi_u = 1 + \mathcal{K}\eta$.



If $\eta(x)$ is peaked with the 120^o angle, then $\eta(u)$ is cusped with the singularity of the type

$$\eta(u) = \frac{c^2}{2} - A|u|^{2/3} + o(|u|^{2/3}).$$

G. Stokes (1880), J. Toland (1978), P. Plotnikov (1982).

Periodic solutions with the peaked profile

What singular solutions are admitted in Babenko's equation and for what values of c?

$$\boxed{c^2 \mathcal{K}\eta - \eta = \frac{1}{2}\mathcal{K}\eta^2 + \eta\mathcal{K}\eta,}$$

Near the singular solutions, it makes sense to use $\eta(u) = \frac{c^2}{2} - \tilde{\eta}(u)$ with $\tilde{\eta}$ satisfying the fixed-point equation

$$\tilde{\eta} = T_c(\tilde{\eta}) := \frac{c^2}{2} + \frac{1}{2}\mathcal{K}(\tilde{\eta}^2) + \tilde{\eta}\mathcal{K}\tilde{\eta},$$

where $\tilde{\eta}(0) = 0$.

Periodic solutions with the peaked profile

Theorem (Locke–P, Appl. Math. Lett. 161 (2025) 109359)

If the solution of $\tilde{\eta} = T_c(\tilde{\eta})$ is singular at $u = 0$ with the singularity of the type

$$\tilde{\eta}(u) = A|u|^\alpha + \mathcal{O}(|u|^{2\alpha}), \quad \alpha \in (0, 1],$$

with some $A > 0$, then necessarily, $\alpha = \frac{2}{3}$.

- Parameters c and A are not defined by the local expansion.
- Some arguments about the unique definition of $c = c_*$ can be found for the model equations (fractional KdV, Whitham):

  J. Dahne, J. Diff. Eqs. 401 (2024) 550

  M. Ehrnström, O.I.H. Mæhlen, K. Varholm, Ann. Inst. H. Poincaré C (2025)
- The proof of uniqueness of c_* is open in Babenko's equation. Numerical continuations suggest $c_* \approx 1.0922$ but cannot approach c_* due to the slow convergence of Fourier series.

Periodic solutions with the peaked profile

Theorem (Locke–P, Appl. Math. Lett. 161 (2025) 109359)

If the solution of $\tilde{\eta} = T_c(\tilde{\eta})$ *is singular at* $u = 0$ *with the singularity of the type*

$$\tilde{\eta}(u) = A|u|^{2/3} + B|u|^{\beta} + \mathcal{O}(|u|^{2/3+\beta}), \quad \beta \in \left(\frac{2}{3}, 2\right),$$

with some $A > 0$ *and* $B \neq 0$*, then necessarily,* $\beta \approx 1.46$ *is a root of the transcendental equation*

$$\left(\beta + \frac{2}{3}\right)\cot\left(\frac{\pi}{2}(\beta - \frac{1}{3})\right) - \beta\tan\left(\frac{\pi\beta}{2}\right) = \frac{2}{\sqrt{3}}.$$

These results recover the asymptotic results from

M.A. Grant, J. Fluid Mech. 59 (1973) 257

Ideas of the proof

The singular behavior of the two quadratic terms in

$$\boxed{\tilde{\eta} = T_c(\tilde{\eta}) := \frac{c^2}{2} + \frac{1}{2}\mathcal{K}(\tilde{\eta}^2) + \tilde{\eta}\mathcal{K}\tilde{\eta}.}$$

is different for solution of the form $\tilde{\eta}(u) = A|u|^\alpha + \mathrm{o}(|u|^\alpha)$ as $|u| \to 0$:

$$\mathcal{K}|u|^\alpha = -\alpha\mathcal{H}(|u|^{\alpha-1}\mathrm{sgn}(u)) = -\alpha \tan\left(\frac{\pi\alpha}{2}\right)|u|^{\alpha-1} + \mathcal{O}(1),$$
$$\mathcal{K}|u|^{2\alpha} = -2\alpha\mathcal{H}(|u|^{2\alpha-1}\mathrm{sgn}(u)) = -2\alpha\tan(\pi\alpha)|u|^{2\alpha-1} + \mathcal{O}(1).$$

This yields

$$T_c\tilde{\eta} = -A^2\alpha\left[\tan(\pi\alpha) + \tan\left(\frac{\pi\alpha}{2}\right)\right]|u|^{2\alpha-1} + \mathcal{O}(1) + \mathrm{o}(|u|^{2\alpha-1}).$$

Since $2\alpha - 1 < \alpha$ for $\alpha \in (0,1)$, the fixed-point equation $\tilde{\eta} = T_c\tilde{\eta}$ can be satified as $|u| \to 0$ only if

$$\tan(\pi\alpha) + \tan\left(\frac{\pi\alpha}{2}\right) = 0,$$

which has the only root $\frac{2}{3}$ for α in $(0,1)$.

4. Stability criterion for co-periodic perturbations

Assume that the traveling wave with profile $\eta \in H^\infty_{\rm per}(\mathbb{T})$ exists for this $c \in (1, c_{\max})$. Consider solutions $(\hat{v}, \hat{w}) \in H^1_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T})$ of the spectral stability problem for some values of $\lambda \in \mathbb{C}$:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

where

- $\mathcal{H}$, $\mathcal{M} = 1 + \mathcal{K}\eta + \eta'\mathcal{H}$, and $\mathcal{M}^* = 1 + \mathcal{K}\eta - \mathcal{H}(\eta' \cdot)$ are bounded operators in $L^2(\mathbb{T})$.
- $\mathcal{K} = -\mathcal{H}\partial_u$ and $\mathcal{L}$ are unbounded operators in $L^2(\mathbb{T})$ with the domain in $H^1_{\rm per}(\mathbb{T})$. The spectrum of λ is purely discrete.
- Moreover, $\mathcal{L}$ is the linearized Babenko's operator:

$$\mathcal{L} = c^2\mathcal{K} - (1 + \mathcal{K}\eta) - \eta\mathcal{K} - \mathcal{K}(\eta \cdot)$$

so that $\mathcal{L}\eta' = 0$ and $\mathcal{L}\partial_c\eta = -2c\mathcal{K}\eta$.

What determine new unstable EV (eigenvalue) λ?

For the linearized Babenko's operator $\mathcal{L} : H^1_{\rm per}(\mathbb{T}) \to L^2(\mathbb{T})$, the spectrum is known at $c = 1$, where $\eta = 0$:

$$c = 1 : \quad \sigma(\mathcal{L}) = \{|n| - 1, \quad n \in \mathbb{Z}\} = \{-1, 0, 1, 2, \dots\}.$$

By using the small-amplitude expansion with $c^2 = 1 + a^2 + \mathcal{O}(a^4)$,

$$\eta(u) = a\cos(u) + a^2 \left[\cos(2u) - \frac{1}{2}\right] + \frac{3}{2}a^3\cos(3u) + \mathcal{O}(a^4),$$

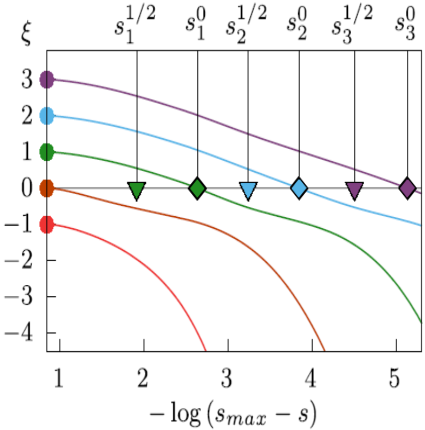
one can show that the zero EV splits into a zero eigenvalue associated with the odd eigenfunction $\eta'(u)$ and a small negative EV $-2a^2 + \mathcal{O}(a^4)$ associated with an even eigenfunction.

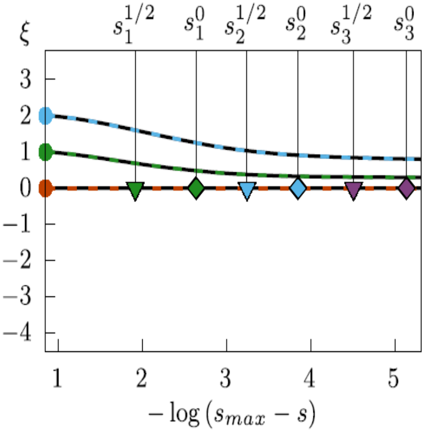
Similarly, all double positive eigenvalues split for even and odd eigenfunctions and may cross 0 for larger values of a ($c > 1$).

What determine new unstable EV (eigenvalue) λ?

Numerical results from

S. Dyachenko, A. Semenova, J. Comp. Phys. 492 (2023) 112411





What determine new unstable EV (eigenvalue) λ?

Q: Do unstable eigenvalues bifurcate when the linearized Babenko's operator admit zero EV?

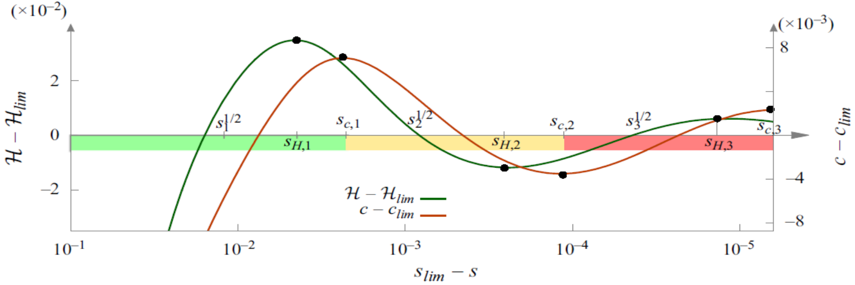
$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

$$c \neq c_0 : \quad \mathrm{Ker}(\mathcal{K}) = \mathrm{span}(1), \quad \mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta'),$$
$$c = c_0 : \quad \mathrm{Ker}(\mathcal{K}) = \mathrm{span}(1), \quad \mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta', v_0),$$

What determine new unstable EV (eigenvalue) λ?

A: No. They bifurcate at the extremal points of energy!

$$H(\eta) = \oint \left[\eta^2(1 + \mathcal{K}\eta) + c^2\eta\mathcal{K}\eta\right] du, \quad \phi = -c\mathcal{H}\eta.$$



M. Tanaka, J. Phys. Soc. Japan 52 (1983) 3047; J. Fluid Mech. 156 (1985) 281

P. G. Saffman, J. Fluid Mech. 159 (1985) 169

What determine new unstable EV (eigenvalue) λ?

Babenko's equation $c^2\mathcal{K}\eta - \eta = \frac{1}{2}\mathcal{K}\eta^2 + \eta\mathcal{K}\eta$ is the Euler–Lagrange equation of the action functional

$$\Lambda_c(\eta) := \frac{1}{2}\langle (c^2 K - 1)\eta, \eta\rangle - \frac{1}{2}\langle K\eta^2, \eta\rangle, \quad \eta \in H^1_{\rm per}(\mathbb{T}),$$

so that

$$\frac{d}{dc}\Lambda_c(\eta) = c\langle K\eta, \eta\rangle + \langle (c^2 K\eta - \eta - \frac{1}{2}K\eta^2 - \eta K\eta), \partial_c \eta\rangle = P(\eta),$$

where $P(\eta) = c\langle \mathcal{K}\eta, \eta\rangle$ is the horizontal momentum. This yields $\frac{d}{dc}H(\eta) = c\frac{d}{dc}P(\eta)$ since $\Lambda_c(\eta) = cP(\eta) - H(\eta)$, we obtain

$$\Lambda_c(\eta) = cP(\eta) - H(\eta) \quad \Rightarrow \frac{d}{dc}\Lambda_c(\eta) = c\frac{d}{dc}P(\eta) - \frac{d}{dc}H(\eta) + P(\eta).$$

Extremal points of energy coincide with extremal points of the horizontal momentum, as in the BO or KdV equations.

Bifurcations of unstable eigenvalues

Theorem (Dyachenko–P, Physica D (2025))

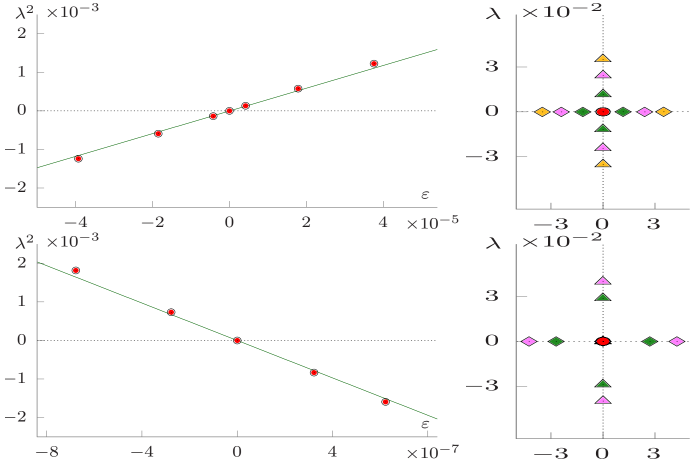
Assume the existence of $\eta \in C^\infty_{\rm per}(\mathbb{T})$ *with* $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta')$. *The generalized null space of the spectral problem is at least six-dimensional if and only if* $\frac{d}{dc}P(\eta)|_{c=c_0} = 0$. *Moreover, new eigenvalues* λ *satisfy*

$$\mathcal{B}\lambda^2 + (c-c_0)\frac{d^2}{dc^2}P(\eta)|_{c=c_0} + \mathcal{O}((c-c_0)^2) = 0,$$

for some $\mathcal{B} > 0$.

- The numerical coefficient $\mathcal{B} > 0$ is approximated numerically.
- Since $c - c_0$ and $\frac{d^2}{dc^2}P(\eta)|_{c=c_0}$ alternate between each extremal point, the new unstable eigenvalues always bifurcate in the direction of increasing steepness.

Bifurcations of unstable eigenvalues



Jordan block computations

We have the spectral stability problem:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

By the assumption, $\mathrm{Ker}(\mathcal{K}) = \mathrm{span}(1)$ and $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta')$. For every $\lambda \neq 0$, we have two orthogonality conditions:

$$\langle 1, \mathcal{M}v \rangle = \langle 1 + 2\mathcal{K}\eta, v \rangle = 0,$$
$$\langle \eta', -2c\mathcal{H}v + \mathcal{M}^* w \rangle = -2c\langle \mathcal{K}\eta, v \rangle + \langle \eta', w \rangle = 0.$$

These constraints determine the size of the Jordan block at $\lambda = 0$.

Jordan block computations

We have the spectral stability problem:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

Kernel at $\lambda = 0$:

$$\left(\begin{array}{c} v \\ w \end{array} \right) = a_1 \left(\begin{array}{c} \eta' \\ 0 \end{array} \right) + a_2 \left(\begin{array}{c} 0 \\ 1 \end{array} \right),$$

where (a_1, a_2) are arbitrary.

Jordan block computations

We have the spectral stability problem:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

First generalized kernel at $\lambda = 0$:

$$\left\{ \begin{array}{ll} \mathcal{K} w_1 & = a_1 \mathcal{M} \eta', \\ \mathcal{L} v_1 & = -2c a_1 \mathcal{H} \eta' + a_2 \mathcal{M}^* 1. \end{array} \right.$$

There exists a solution $(v_1, w_1) \in H^1_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T})$:

$$\left(\begin{array}{c} v \\ w \end{array} \right) = a_1 \left(\begin{array}{c} -\partial_c \eta \\ \mathcal{H}\eta \end{array} \right) + a_2 \left(\begin{array}{c} -1 \\ 0 \end{array} \right).$$

Jordan block computations

We have the spectral stability problem:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

Second generalized kernel at $\lambda = 0$:

$$\left\{ \begin{array}{ll} \mathcal{K} w_2 & = -a_1 \mathcal{M} \partial_c \eta - a_2 \mathcal{M} 1, \\ \mathcal{L} v_2 & = 2ca_1 \mathcal{H} \partial_c \eta + a_1 \mathcal{M}^* \mathcal{H} \eta. \end{array} \right.$$

We have $\langle 1, \mathcal{M} \partial_c \eta \rangle = 0$ due to

$$\langle 1 + \mathcal{K}\eta, \eta \rangle = 0 \quad \Rightarrow \quad \langle 1, \mathcal{M} \partial_c \eta \rangle = \langle 1 + 2\mathcal{K}\eta, \partial_c \eta \rangle = 0.$$

But $\langle 1, \mathcal{M} 1 \rangle = 1 \neq 0$. Similarly, $\langle \eta', 2c\mathcal{H}\partial_c \eta + \mathcal{M}^* \mathcal{H} \eta \rangle = \frac{d}{dc} P(\eta)$.

Jordan block computations

We have the spectral stability problem:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

The Jordan canonical forms are

$$\frac{d}{dc}P(\eta) \neq 0 : \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\frac{d}{dc}P(\eta) = 0, \quad \mathcal{B} \neq 0 : \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Jordan block computations

We have the spectral stability problem:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

If $\frac{d}{dc}P(\eta) = 0$, the Jordan chain of eigenvectors for $(a_1, a_2) = (1, 0)$ is

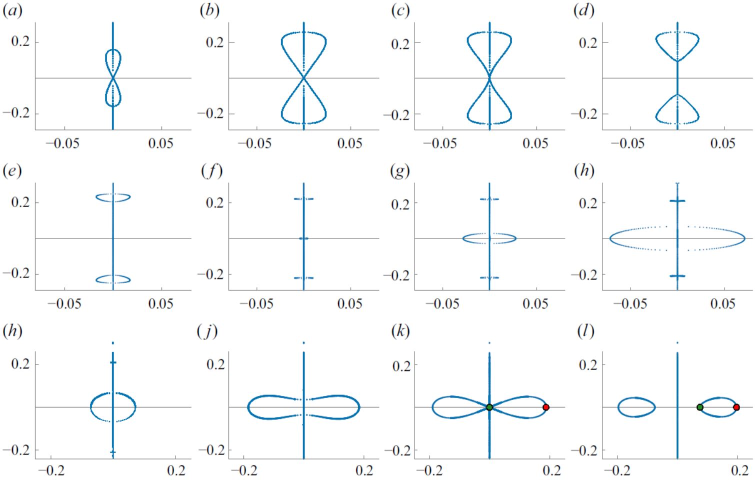
$$\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} \eta' \\ 0 \end{pmatrix}, \quad \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} -\partial_c \eta \\ \mathcal{H}\eta \end{pmatrix}, \quad \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}, \quad \begin{pmatrix} v_3 \\ w_3 \end{pmatrix},$$

where

$$\left\{ \begin{array}{ll} \mathcal{K} w_2 & = -\mathcal{M}\partial_c \eta, \\ \mathcal{L} v_2 & = 2c\mathcal{H}\partial_c \eta + \mathcal{M}^* \mathcal{H}\eta. \end{array} \right. \qquad \left\{ \begin{array}{ll} \mathcal{K} w_3 & = \mathcal{M} v_2, \\ \mathcal{L} v_3 & = -2c\mathcal{H} v_2 + \mathcal{M}^* w_2. \end{array} \right.$$

and $\mathcal{B} := \langle \eta', -2c\mathcal{H} v_3 + \mathcal{M}^* w_3 \rangle$ with $\mathcal{B} > 0$ (numerically).
Justification of the normal form is done by Puiseux expansions.

5. Bifurcations of the modulational stability bands



B. Deconinck, S. Dyachenko, A. Semenova, J. Fluid Mech. 995 (2024) A2

Stability with respect to arbitrary perturbations

We consider solutions $(v, w) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ of

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}.$$

By the Floquet–Bloch theory, we can use the integral transform

$$v(u) = \int_{\mathbb{B}} \hat{v}(u, \mu) e^{i\mu u} d\mu, \quad w(u) = \int_{\mathbb{B}} \hat{w}(u, \mu) e^{i\mu u} d\mu, \quad \mathbb{B} = \left(-\frac{1}{2}, \frac{1}{2}\right]$$

and consider solutions $(\hat{v}, \hat{w}) \in H^1_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T})$ of

$$\begin{pmatrix} \mathcal{L}_\mu & 0 \\ 0 & \mathcal{K}_\mu \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H}_\mu & \mathcal{M}^*_\mu \\ \mathcal{M}_\mu & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

where μ-dependent operators obtained from $\mathcal{L}_\mu = e^{-i\mu u} \mathcal{L} e^{i\mu u}$.
The spectrum is still purely discrete for every $\mu \in \mathbb{B}$.

Stability with respect to arbitrary perturbations

$\underline{\mathcal{H}_\mu = e^{-i\mu x}\mathcal{H}e^{i\mu x}}$:

Since $\mathcal{H}\sum_{n\in\mathbb{Z}}\hat{f}_n e^{i(\mu+n)u} = i\sum_{n\in\mathbb{Z}} \mathrm{sgn}(\mu+n)\hat{f}_n e^{i(\mu+n)u}$, we get

$$\mathcal{H}_\mu f = \mathcal{H}f \pm i\langle 1, f\rangle =: \mathcal{H}_\pm, \quad \pm\mu > 0.$$

Stability with respect to arbitrary perturbations

$\underline{\mathcal{H}_\mu = e^{-i\mu x}\mathcal{H}e^{i\mu x}}$:

Since $\mathcal{H}\sum_{n\in\mathbb{Z}}\hat{f}_n e^{i(\mu+n)u} = i\sum_{n\in\mathbb{Z}}\text{sgn}(\mu+n)\hat{f}_n e^{i(\mu+n)u}$, we get

$$\mathcal{H}_\mu f = \mathcal{H}f \pm i\langle 1, f\rangle =: \mathcal{H}_\pm, \quad \pm\mu > 0.$$

$\underline{\mathcal{K}_\mu = e^{-i\mu x}\mathcal{K}e^{i\mu x}}$:

Since $\mathcal{K}_\mu = -\mathcal{H}_\mu(\partial_u + i\mu)$, we get

$$\mathcal{K}_\mu = \mathcal{K} - i\mu\mathcal{H}_\pm.$$

Stability with respect to arbitrary perturbations

$\underline{\mathcal{H}_\mu = e^{-i\mu x}\mathcal{H}e^{i\mu x}:}$

Since $\mathcal{H}\sum_{n\in\mathbb{Z}}\hat{f}_n e^{i(\mu+n)u} = i\sum_{n\in\mathbb{Z}}\text{sgn}(\mu+n)\hat{f}_n e^{i(\mu+n)u}$, we get

$$\mathcal{H}_\mu f = \mathcal{H}f \pm i\langle 1, f\rangle =: \mathcal{H}_\pm, \quad \pm\mu > 0.$$

$\underline{\mathcal{K}_\mu = e^{-i\mu x}\mathcal{K}e^{i\mu x}:}$

Since $\mathcal{K}_\mu = -\mathcal{H}_\mu(\partial_u + i\mu)$, we get

$$\mathcal{K}_\mu = \mathcal{K} - i\mu\mathcal{H}_\pm.$$

$\underline{\mathcal{M}_\mu = e^{-i\mu x}\mathcal{M}e^{i\mu x}:}$

Similarly,

$$\mathcal{M}_\mu = 1 + \mathcal{K}\eta + \eta'\mathcal{H}_\pm,$$
$$\mathcal{M}_\mu^* = 1 + \mathcal{K}\eta - \mathcal{H}_\pm(\eta'\cdot).$$

Stability with respect to arbitrary perturbations

$\mathcal{L}_\mu = e^{-i\mu x}\mathcal{L}e^{i\mu x}$:

Since $\mathcal{L} = c^2\mathcal{K} - (1+\mathcal{K}\eta) - \eta\mathcal{K} - \mathcal{K}(\eta\,\cdot)$, we also have

$$\mathcal{L}_\mu = \mathcal{L} - i\mu\mathcal{L}_\pm,$$

where

$$\mathcal{L}_\pm = c^2\mathcal{H}_\pm - \eta\mathcal{H}_\pm - \mathcal{H}_\pm(\eta\cdot).$$

Hence, the spectral stability problem becomes a eigenvalue problem in λ with linear terms in μ:

$$\left\{ \begin{array}{ll} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}), \end{array} \right.$$

where $\pm\mu > 0$.

Stability with respect to arbitrary perturbations

In the limit $\mu \to 0^{\pm}$, we get

$$\begin{cases} \mathcal{K}\hat{w} & = \lambda \mathcal{M}_{\pm} \hat{v}, \\ \mathcal{L}\hat{v} & = \lambda (\mathcal{M}_{\pm}^* \hat{w} - 2c\mathcal{H}_{\pm} \hat{v}), \end{cases}$$

or explicitly

$$\begin{cases} \mathcal{K}\hat{w} & = \lambda \left(\mathcal{M}\hat{v} \pm i\langle 1, \hat{v} \rangle \eta' \right), \\ \mathcal{L}\hat{v} & = \lambda \left(\mathcal{M}^* \hat{w} - 2c\mathcal{H}\hat{v} \mp ci\langle 1, \hat{v} \rangle \right). \end{cases}$$

All projection terms are removed by the transformation

$$\hat{v} = \mp i\langle 1, v \rangle \eta' + v, \qquad \hat{w} = \pm ic\langle 1, v \rangle + w. \tag{1}$$

where $(v, w) \in H^1_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T})$ solves the previous stability problem for co-periodic perturbations.

Bifurcation of the modulational stability band

Theorem (Dyachenko–Marangell–P, in progress (2025))

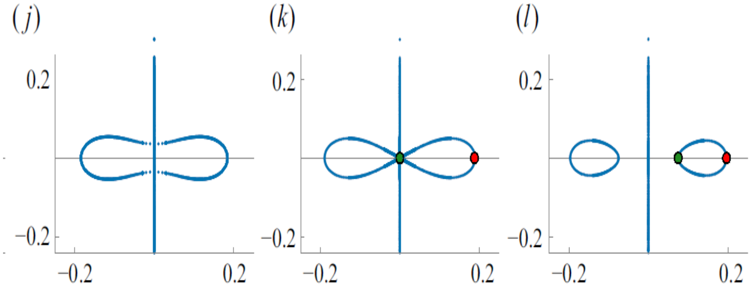
Assume the existence of $\eta \in C^\infty_{\rm per}(\mathbb{T})$ *with* $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta')$. *Assume* $\frac{d}{dc}P(\eta)|_{c=c_0} = 0$ *and* $\mathcal{B} \neq 0$. *The modulational stability bands* $(\mu, \lambda(\mu))$ *near* $(0,0)$ *satisfy*

$$\lambda^2 \left(\mathcal{B}\lambda^2 + (c - c_0)\frac{d^2}{dc^2}P(\eta)|_{c=c_0} \right) - i\mu\lambda \frac{d}{dc}\|\eta\|^2_{L^2} = \mathcal{O}(\lambda^6 + \mu^2).$$

- For $c = c_0$, three bands exist near $(0,0)$ from solutions of the cubic equation $\mathcal{B}\lambda^3 - i\mu\frac{d}{dc}\|\eta\|^2_{L^2} = 0$.
- For $c \neq c_0$, the three bands split differently along $i\mathbb{R}$ and $\mathbb{R}$.

$$c\frac{d}{dc}P(\eta) - \frac{d}{dc}\|\eta\|^2_{L^2} = 5P(\eta) \quad \Rightarrow \quad \frac{d}{dc}\|\eta\|^2_{L^2} < 0.$$

Bifurcation of the modulational stability band



Jordan block computations

We have the spectral stability problem:

$$\begin{cases} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}), \end{cases}$$

where $\pm\mu > 0$.

Projections near $(\mu, \lambda) = (0, 0)$ are consistent under the assumptions if

$$\mu = \epsilon^3\mu_1, \quad \lambda = \epsilon\lambda_1, \quad c - c_0 = \epsilon^2 c_1,$$

where μ_1, λ_1, c_1 are nonzero in the limit $\epsilon \to 0$.

Jordan block computations

We have the spectral stability problem:

$$\begin{cases} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}), \end{cases}$$

where $\pm\mu > 0$.

The asymptotic expansion of solutions $(\hat{v}, \hat{w}) \in H^1_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T})$ for given scaled values of (μ, λ) is

$$\begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \begin{pmatrix} \eta' \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} \hat{v}_1 \\ \hat{w}_1 \end{pmatrix} + \epsilon^2 \begin{pmatrix} \hat{v}_2 \\ \hat{w}_2 \end{pmatrix} + \epsilon^3 \begin{pmatrix} \hat{v}_3 \\ \hat{w}_3 \end{pmatrix} + \epsilon^4 \begin{pmatrix} \hat{v}_4 \\ \hat{w}_4 \end{pmatrix} + \mathcal{O}(\epsilon^5),$$

Jordan block computations

We have the spectral stability problem:

$$\left\{ \begin{array}{ll} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}), \end{array} \right.$$

where $\pm\mu > 0$.

First generalized kernel with the additional projection terms:

$$\left(\begin{array}{c} \hat{v}_1 \\ \hat{w}_1 \end{array} \right) = \lambda_1 \left[\left(\begin{array}{c} -\partial_c\eta \\ \mathcal{H}\eta \end{array} \right) \pm i\langle 1, \partial_c\eta\rangle \left(\begin{array}{c} \eta' \\ -c \end{array} \right) \right].$$

Jordan block computations

We have the spectral stability problem:

$$\begin{cases} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda \mathcal{M}_\pm \hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^* \hat{w} - 2c\mathcal{H}_\pm \hat{v}), \end{cases}$$

where $\pm\mu > 0$.

Second generalized kernel at $\lambda = 0$:

$$\begin{pmatrix} \hat{v}_2 \\ \hat{w}_2 \end{pmatrix} = \lambda_1^2 \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} + a \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

where $a \in \mathbb{R}$ is to be determined.

Jordan block computations

We have the spectral stability problem:

$$\begin{cases} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}), \end{cases}$$

where $\pm\mu > 0$.

Third generalized kernel at $\lambda = 0$:

$$\begin{pmatrix} \hat{v}_3 \\ \hat{w}_3 \end{pmatrix} = \lambda_1^3 \left[\begin{pmatrix} v_3 \\ w_3 \end{pmatrix} \mp i\langle 1, v_3\rangle \begin{pmatrix} \eta' \\ -c \end{pmatrix} \right] + i\mu_1 \begin{pmatrix} \frac{c}{2}\partial_c\eta - \eta \\ 0 \end{pmatrix} - a\lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Jordan block computations

We have the spectral stability problem:

$$\left\{ \begin{array}{ll} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}), \end{array} \right.$$

where $\pm\mu > 0$.

At the order of $\mathcal{O}(\epsilon^4)$, we get

$$\left\{ \begin{array}{ll} \mathcal{K}\hat{w}_4 & = \lambda_1\mathcal{M}_\pm\hat{v}_3 + i\mu_1\mathcal{H}_\pm\hat{w}_1, \\ \mathcal{L}\hat{v}_4 & = \lambda_1\left(\mathcal{M}_\pm^*\hat{w}_3 - 2c\mathcal{H}_\pm\hat{v}_3\right) + i\mu_1\mathcal{L}_\pm\hat{v}_1. \end{array} \right.$$

Fredholm condition for $\mathrm{Ker}(\mathcal{K})$ yields $a = \lambda_1^2\langle(1 + 2\mathcal{K}\eta), v_3\rangle$.
Fredholm condition for $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta')$ yields

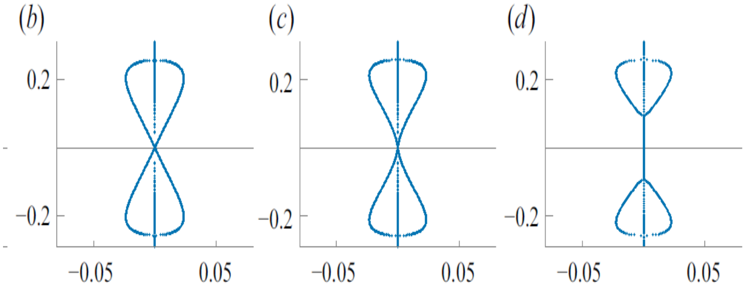
$$\mathcal{B}\lambda_1^4 + c_1\lambda_1^2\frac{d^2}{dc^2}P(\eta)|_{c=c_0} - i\mu_1\lambda_1\frac{d}{dc}\|\eta\|_{L^2}^2 = 0.$$

Other bifurcations of the modulational stability bands

Breakdown of the figure-8 appears if $\eta \in C^\infty_{\rm per}(\mathbb{T})$ exists with $\text{Ker}(\mathcal{L}) = \text{span}(\eta')$ and $\frac{d}{dc}P(\eta)|_{c=c_0} \neq 0$, whereas

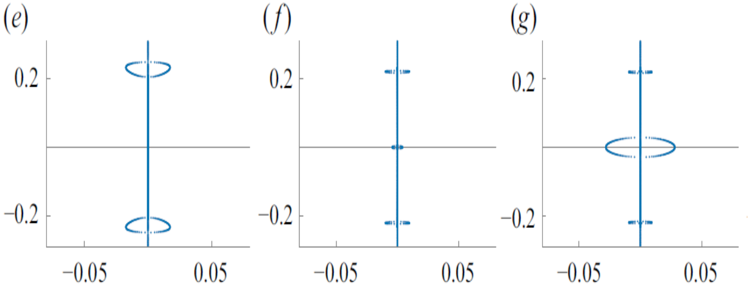
$$\Delta(c) := 4\mathcal{P}'(c)\mathcal{N}(c) - 5\mathcal{P}(c)\mathcal{N}'(c) = 0,$$

where $\mathcal{P}(c) = P(\eta)$ and $\mathcal{N}(c) = \|\eta\|^2_{L^2}$.



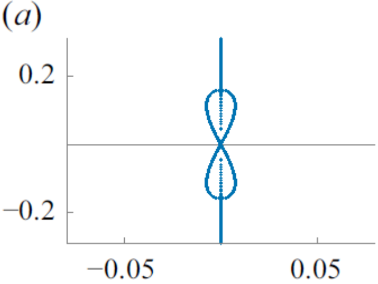
Other bifurcations of the modulational stability bands

New instability bubble appears if $\eta \in C^\infty_{\rm per}(\mathbb{T})$ exists with $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\psi_0)$ for anti-periodic perturbations with $\mu = \frac{1}{2}$.



Other bifurcations of the modulational stability bands

New figure-8 appears if $\eta \in C^{\infty}_{\rm per}(\mathbb{T})$ exists with $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta', v_0)$ for co-periodic perturbations with $\mu = 0$.



Summary

- The full water wave equations are replaced by two evolution (Benjamin–Ono-type) equations by conformal transformations.
- Existence of traveling waves with both smooth and peaked profiles is obtained from smooth and singular solutions of the scalar pseudo–diferential (Babenko) equation.
- Stability of traveling waves with smooth profiles is obtained from a purely discrete spectrum of the spectral problem.
- The (infinite) recurrence of co-periodic bifurcations, fold bifurcations, vertical slope bifurcations, and anti-periodic bifurcations is a conjecture to be explained (snaking behavior).