Polynomials

A **polynomial** of **degree** n is a function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_0, \ldots, a_n$ are real numbers (called **coefficients**) and n is a positive integer (called the **degree** of $p(x)$).

Polynomials can be represented by the vector of their coefficients in a vector space,

$$\mathbf{u} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbf{R}^{n+1}$$

Linear operations

Linear operations are defined on polynomials of the same (largest) degree, e.g.

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$
$$q(x) = b_0 + b_1 x + \cdots + b_n x^n$$

- Addition of $p(x)$ and $q(x)$

$$p(x) + q(x) := (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

- Scalar multiplication of $p(x)$ by $k \in \mathbf{R}$:

$$kp(x) := (ka_0) + (ka_1)x + \cdots + (ka_n)x^n.$$

Example

$$p(x) = 1 + x^2, \qquad q(x) = 3 - 2x$$

Theorem

Let P_n be a set of all polynomials of degree n and smaller. Then, P_n is a vector space such that if $p(x) \in P_n$ then $p(x)$ is uniquely represented by the basic functions $\{1, x, x^2, ..., x^n\}$. Dimension of P_n is $n + 1$.

Various basis in P_n are possible. On the other hand, spanning set of functions may not be linearly independent.

Example

Consider two subspaces of polynomials

$$U = \text{span}\{1 + x^2, x + x^3, 1 - x^2, x^3\}$$

$$V = \text{span}\{1, 1 + x^2, 2 - 3x^2\}$$

Theorem Let $\{p_0(x), p_1(x), ..., p_n(x)\}$ be polynomials in P_n of degrees $0, 1, ..., n$. Then, the set of polynomials $\{p_0(x), p_1(x), ..., p_n(x)\}$ is a basis in P_n.

Example

Let $p_1(x), p_2(x), p_3(x), p_4(x)$ be polynomials of degree at most two. Show that at least one polynomial is linearly dependent of the others.

Example

Consider a subspace of all polynomials of degree n with a root at $x = 2$, such that

$$U = \{p(x) \in P_n : \; p(2) = 0\}$$

Find the basis of vectors for U.