Polynomials

A polynomial of degree $n$ is a function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_0, \ldots, a_n$ are real numbers (called coefficients) and $n$ is a positive integer (called the degree of $p(x)$).

Polynomials can be represented by the vector of their coefficients in a vector space,

$$u = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^{n+1}$$
Linear operations

Linear operations are defined on polynomials of the same (largest) degree, e.g.

\[ p(x) = a_0 + a_1 x + \cdots + a_n x^n \]
\[ q(x) = b_0 + b_1 x + \cdots + b_n x^n \]

- Addition of \( p(x) \) and \( q(x) \)

\[ p(x) + q(x) : = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \]

- Scalar multiplication of \( p(x) \) by \( k \in \mathbb{R} \):

\[ kp(x) : = (ka_0) + (ka_1)x + \cdots + (ka_n)x^n. \]

Example

\[ p(x) = 1 + x^2, \quad q(x) = 3 - 2x \]
Theorem
Let $P_n$ be a set of all polynomials of degree $n$ and smaller. Then, $P_n$ is a vector space such that if $p(x) \in P_n$ then $p(x)$ is uniquely represented by the basic functions $\{1, x, x^2, \ldots, x^n\}$. Dimension of $P_n$ is $n + 1$.

Various basis in $P_n$ are possible. On the other hand, spanning set of functions may not be linearly independent.

Example
Consider two subspaces of polynomials

$$U = \text{span}\{1 + x^2, x + x^3, 1 - x^2, x^3\}$$

$$V = \text{span}\{1, 1 + x^2, 2 - 3x^2\}$$
Theorem  Let \( \{p_0(x), p_1(x), \ldots, p_n(x)\} \) be polynomials in \( P_n \) of degrees 0, 1, \ldots, \( n \). Then, the set of polynomials \( \{p_0(x), p_1(x), \ldots, p_n(x)\} \) is a basis in \( P_n \).

Example
Let \( p_1(x), p_2(x), p_3(x), p_4(x) \) be polynomials of degree at most two. Show that at least one polynomial is linearly dependent of the others.

Example
Consider a subspace of all polynomials of degree \( n \) with a root at \( x = 2 \), such that

\[
U = \{p(x) \in P_n : p(2) = 0\}
\]

Find the basis of vectors for \( U \).